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On the modulus of continuity for spectral measures in substitution dynamics

Alexander I. Bufetov [a,b,c,d,e], Boris Solomyak [f,*]

[a] *Aix-Marseille Université, CNRS, Centrale Marseille, I2M, UMR 7373, Marseille, France*
[b] *Steklov Institute, Moscow, Russia*
[c] *The Institute for Information Transmission Problems, Moscow, Russia*
[d] *National Research University Higher School of Economics, Moscow, Russia*
[e] *Rice University, Houston, TX, USA*
[f] *Box 354350, Department of Mathematics, University of Washington, Seattle, WA, USA*

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ABSTRACT

The paper gives first quantitative estimates on the modulus of continuity of the spectral measure for weak mixing suspension flows over substitution automorphisms, which yield information about the "fractal" structure of these measures. The main results are, first, a Hölder estimate for the spectral measure of almost all suspension flows with a piecewise constant roof function; second, a log-Hölder estimate for self-similar suspension flows; and, third, a Hölder asymptotic expansion of the spectral measure at zero for such flows. Our second result implies log-Hölder estimates for the spectral measures of translation flows along stable foliations of pseudo-Anosov automorphisms. A key technical tool in the proof of the second result is an "arithmetic-Diophantine" proposition, which has other applications. In Appendix A this proposition is used to derive new decay estimates for the Fourier transforms of Bernoulli convolutions.

* Corresponding author.
E-mail addresses: bufetov@mi.ras.ru (A.I. Bufetov), solomyak@uw.edu (B. Solomyak).

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1. Introduction

Substitution dynamical systems and their spectral properties have been studied for a long time, see [46,26] and references therein. These systems are of intrinsic interest, but also have many links to other areas, both in dynamics and beyond. An incomplete list of these links includes Bratteli–Vershik (adic) transformations [53,54], especially in the stationary case; interval exchange transformations that are periodic for the Rauzy–Veech induction, and translation flows along stable/unstable flows for pseudo-Anosov automorphisms, see [6,7]. Substitutions and associated dynamical systems are also widely used in mathematical physics, in particular, in the study of quasicrystals, see e.g. [2,31,29,44].

The aim of this paper is to estimate the modulus of continuity for the spectral measures of suspension flows over substitution dynamical systems. Our main assumption is that the substitution matrix has at least two eigenvalues outside the unit circle, which implies that almost every suspension flow, in particular, the self-similar suspension flow, is weak mixing [11,52].

Our first main result is Theorem 4.1 that gives a uniform Hölder bound away from zero for the spectral measure of almost all suspension flows with piecewise constant roof functions. This result does not, however, give specific examples of flows with Hölder spectrum. In the important special case of self-similar suspension flows we are able to obtain log-Hölder estimates on the spectrum; these are contained in our second main result, Theorem 5.1. Our third main result, Theorem 6.2, gives a Hölder asymptotic expansion for the spectral measure of our self-similar suspension flows *at zero*; the Hölder exponent is explicitly computed.

The motivation is to study fine "dimension-like," or "fractal" properties of the spectral measures. Hölder estimates imply lower bounds for the local dimension and Hausdorff dimension of the measures. In mathematical physics (quantum dynamics, discrete Schrödinger operators, etc.) similar investigations have been very active, see e.g. [43,14], but we are not aware of any significant work in this direction in ergodic theory. The hope is that such a study will lead to better understanding of substitution dynamical systems; in particular, there should be a connection with "quantitative rates of weak mixing" (which are conjugacy invariants) in the spirit of [40].

It is worth mentioning that some of our techniques have parallels in the work of Forni and Ulcigrai [27], who proved Lebesgue spectrum of smooth time-changes for the classical horocycle flow on a compact hyperbolic surface using asymptotic properties of "twisted ergodic integrals." Our setting is completely different, but we are also using such integrals in Sections 4 and 5.

The paper is organized as follows. In Section 2 we represent spectral measures of substitution automorphisms by matrix Riesz products; the construction is a generalization of the Riesz product representation of the spectrum for substitutions of constant length, see [46]. In Section 3 we analyze our matrix Riesz products; the main result of this section is Proposition 3.5, an upper bound for the entries of our matrix products. In the following sections, Proposition 3.5 is used to reduce questions about spectral