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Multifractal behavior of polynomial Fourier series ☆

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Abstract

We study the spectrum of singularities of a family of Fourier series with polynomial frequencies, in particular we prove that they are multifractal functions. The case of degree two was treated by S. Jaffard in 1996. Higher degrees require completely different ideas essentially because harmonic analysis techniques (Poisson summation) are useless to study the oscillation at most of the points. We introduce a new approach involving special diophantine approximations with prime power denominators and fine analytic and arithmetic aspects of the estimation of exponential sums to control the Hölder exponent in thin Cantor-like sets.

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Keywords: Spectrum of singularities; Multifractal function; Diophantine approximation; Exponential sums

1. Introduction

The so-called "Riemann's example"

$$R(x) = \sum_{n=1}^{\infty} \frac{\sin(2\pi n^2 x)}{n^2}$$

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has a long and fascinating history that has generated a vast literature (see [2], [6, §1], [7] and references). Just to give a glimpse of it in few words, we mention that according to Weierstrass [35,8], Riemann considered R to be an example of a continuous nowhere differentiable function. Indeed G.H. Hardy [14] proved in 1916 that R is not differentiable at any irrational value and at some families of rational values. In 1970, J. Gerver [11] proved, when he was a student, that R is differentiable at infinitely many rationals. The combination of [14] and [11] gives a full characterization of the differentiability points of R.

More recently several authors have shown interest on the global properties of R and allied functions (this interest was initially linked to wavelet methods [21,16]). For instance, it is known that the (box counting) dimension of the graph of R is $5/4$, in particular it is a fractal [3,4].

S. Jaffard [19] proved that R is a *multifractal function*, meaning that if we classify the points in $[0,1]$ according to the Hölder exponent of R, in the resulting sets we find infinitely many distinct Hausdorff dimensions. The terminology, introduced firstly in the context of turbulent fluid mechanics (see [1]), suggests that a multifractal object is a fractal set with an intricate structure, containing fractal subsets of different dimensions at different scales.

The multifractal nature of a continuous function $f:[0,1]\to\mathbb{C}$ is represented by its *spectrum of singularities*

$$d_f(\beta)=\dim_H\big\{x\colon\ \beta_f(x)=\beta\big\}$$

where $\dim_H$ denotes the Hausdorff dimension and $\beta_f(x)$ is the Hölder exponent of f at x given by

$$\beta_f(x)=\sup\big\{\gamma\colon\ f\in C^\gamma(x)\big\}$$

with

$$C^\gamma(x)=\big\{f\colon\ \big|f(x+h)-P(h)\big|=O\big(|h|^\gamma\big),\ \text{for some } P\in\mathbb{C}[X],\ \deg P\leqslant\gamma\big\}.$$

For $\beta_f(x)\leqslant 1$ we have the simpler and usual definition

$$\beta_f(x)=\sup\big\{\gamma\leqslant 1\colon\ \big|f(x+h)-f(x)\big|=O\big(|h|^\gamma\big)\big\}.$$

We let $d_f(\beta)$ undefined if $\{x\colon\ \beta_f(x)=\beta\}$ is the empty set. In this way, the domain of d_f is always a subset of $[0,\infty)$.

For a “purely fractal” function as the celebrated Weierstrass nondifferentiable function, the graph of d_f consists of a finite number of points while for a multifractal function we observe a non-discrete graph [20].

The main results in [19] are summarized saying that for a given $\alpha>1$ the following function (already appearing in early works of Hardy and Littlewood [15])

$$R_\alpha(x)=\sum_{n=1}^{\infty}\frac{\sin(2\pi n^2x)}{n^\alpha}$$