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A relative higher index theorem, diffeomorphisms and positive scalar curvature

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Abstract

We prove a general relative higher index theorem for complete manifolds with positive scalar curvature towards infinity. We apply this theorem to study Riemannian metrics of positive scalar curvature on manifolds. For every two metrics of positive scalar curvature on a closed manifold and a Galois cover of the manifold, we define a secondary higher index class. Non-vanishing of this higher index class is an obstruction for the two metrics to be in the same connected component of the space of metrics of positive scalar curvature. In the special case where one metric is induced from the other by a diffeomorphism of the manifold, we obtain a formula for computing this higher index class. In particular, it follows that the higher index class lies in the image of the Baum–Connes assembly map.

Keywords: Higher index theory; Baum–Connes conjecture; K-theory; Group C^*-algebras; Positive scalar curvature; Diffeomorphism

1. Introduction

In this paper, we use methods from noncommutative geometry to study problems of positive scalar curvature on manifolds. From the work of Miščenko [21], Kasparov [15], and Connes and Moscovici [9], methods from noncommutative geometry have found many impressive

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applications towards geometry and topology, in particular, to those related to the Novikov conjecture and the positive scalar curvature problem. The fact that the positive scalar curvature problem is closely related to the Novikov conjecture (or the Baum–Connes conjecture) was already made apparent by Rosenberg in [22]. Block and Weinberger [6], and the second author [24,25] successfully applied noncommutative geometric methods to determine the existence (nonexistence) of positive scalar curvature on certain classes of manifolds. By applying the work of Lott on higher eta invariants (which is noncommutative geometric) [20], Leichtnam and Piazza studied the connectedness of the space of all Riemannian metrics of positive scalar curvature on certain classes of manifolds [18].

One of main tools used in all the studies mentioned above is index theory in the context of noncommutative geometry, often referred to as *higher index theory*. The method of applying (classical) index theory to study the positive scalar curvature problem on manifolds goes back to Lichnerowicz. By applying the Atiyah–Singer index theorem [1], he showed that a compact spin manifold does not support positive scalar curvature metrics if its $\hat{A}$-genus is nonzero [19]. With a refined version of the Atiyah–Singer index theorem [2], Hitchin showed that half of the exotic spheres in dimension 1 and 2 (mod 8) cannot carry metrics of positive scalar curvature [13]. This line of development was pursued further by Gromov and Lawson. In [11], they developed a relative index theorem and obtained nonexistence of positive scalar curvature for a large class of (not necessarily compact) manifolds. In [8], Bunke proved a relative higher index theorem and applied it to study problems of positive scalar curvature on manifolds.

In this paper, we prove a general relative higher index theorem (for both real and complex cases). We apply this theorem to study Riemannian metrics of positive scalar curvature on manifolds. For every two metrics of positive scalar curvature on a closed manifold and a Galois cover of the manifold, there is a naturally defined secondary higher index class. Non-vanishing of this higher index class is an obstruction for the two metrics to be in the same connected component of the space of metrics of positive scalar curvature. In the special case where one metric is induced from the other by a diffeomorphism of the manifold, we obtain a formula for computing this higher index class. In particular, it follows that the higher index class lies in the image of the Baum–Connes assembly map.

It is essential to allow real C^*-algebras and their (real) K-theory groups when studying problems of positive scalar curvature on manifolds, cf. [11,22]. In fact, the (real) K-theory groups of real C^*-algebras provide more refined invariants for obstructions of existence of positive scalar curvature. We point out that the proofs in our paper are written in such a way that they apply to both the real and the complex cases. In order to keep the notation simple, we shall only prove the results for the complex case and indicate how to modify the arguments, if needed, for the real case. From now on, unless otherwise specified, all bundles and algebras are defined over $\mathbb{C}$.

Here is a synopsis of the main results of the paper. Let X_0 and X_1 be two even dimensional[2] spin manifolds with complete Riemannian metrics of positive scalar curvature (uniformly bounded below) away from compact sets. Assume that we have compact subspaces $K_i \subset X_i$ such that there is an (orientation preserving) isometry $\Psi : \Omega_0 \to \Omega_1$, where $\Omega_i \subset X_i - K_i$ is a union of (not necessarily all) connected components of $X_i - K_i$ (see Fig. 1 in Section 4). We emphasize that the Riemannian metric on Ω_i may have *nonpositive* scalar curvature on some compact subset. Let $\mathcal{S}_i$ be the corresponding spinor bundle over X_i. We assume that Ψ lifts to a bundle

[2] In the real case, we assume $\dim X_0 = \dim X_1 \equiv 0 \pmod 8$.