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On the order of indeterminate moment problems

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Abstract

For an indeterminate moment problem we denote the orthonormal polynomials by P_n. We study the relation between the growth of the function $P(z) = (\sum_{n=0}^{\infty} |P_n(z)|^2)^{1/2}$ and summability properties of the sequence $(P_n(z))$. Under certain assumptions on the recurrence coefficients from the three term recurrence relation $zP_n(z) = b_n P_{n+1}(z) + a_n P_n(z) + b_{n-1} P_{n-1}(z)$, we show that the function P is of order α with $0 < \alpha < 1$, if and only if the sequence $(P_n(z))$ is absolutely summable to any power greater than 2α. Furthermore, the order α is equal to the exponent of convergence of the sequence (b_n). Similar results are obtained for logarithmic order and for more general types of slow growth. To prove these results we introduce a concept of an order function and its dual.

We also relate the order of P with the order of certain entire functions defined in terms of the moments or the leading coefficient of P_n.

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1. Introduction and results

Stieltjes discovered the indeterminate moment problem in the memoir [25] from 1894, and one can follow his discoveries in the correspondence with Hermite, cf. [4]. Stieltjes only considered distribution functions on the half-line $[0, \infty)$ corresponding to what is now called the Stieltjes moment problem. It took about 25 years before Hamburger, Nevanlinna and Marcel Riesz laid the foundation of the Hamburger moment problem described by (1). Nevanlinna proved the Nevanlinna parametrization of the full set of solutions to the Hamburger moment problem. Using the four entire functions A, B, C, D, obtained from (3) by letting $n \to \infty$, any solution to the moment problem can be described via a universal parameter space, namely the one-point compactification of the space of Pick functions. Nevanlinna also pointed out what is now called the Nevanlinna extremal solutions corresponding to the degenerate Pick functions, which are a real constant or infinity. Since the same solutions appear in spectral theory for self-adjoint extensions of Jacobi matrices, Simon [24] proposed to call them von Neumann solutions. The classical monographs describing the Nevanlinna parametrization are [1,23,26]. None of these treatises contain a fully calculated example with concrete functions A, B, C, D. Although it was well known that the zeros of B, D interlace and similarly with A, C, nobody seem to have noticed that these functions have the same growth properties before it was done in [5]. In that paper it was proved that the four entire functions A, B, C, D as well as P, Q from Theorem 1.1 have the same order and type called the *order ρ and type τ of the indeterminate moment problem*. Already in 1923 Marcel Riesz had proved the deep result that A, B, C, D are of minimal exponential type, i.e., that $0 \leqslant \rho \leqslant 1$ and if $\rho = 1$, then $\tau = 0$, cf. [1, p. 56].

The first concrete examples, where A, B, C, D were calculated together with a number of solutions, appeared in the beginning of the 1990'ies, see Ismail and Masson [16], Chihara and Ismail [12], Berg and Valent [8]. One source of indeterminate moment problems is q-series, cf. [14], and formulas of Ramanujan as pointed out by Askey [2]. The indeterminate moment problems within the q-Askey scheme were identified by Christiansen in [13]. All these moment problems have order zero, and in Ismail [15] it was conjectured that A, B, C, D should have the same growth properties on a more refined scala than ordinary order. This was proved in [6], by the introduction of a refined scale called logarithmic order and type, so we can speak about logarithmic order $\rho^{[1]}$ and logarithmic type $\tau^{[1]}$ of a moment problem of order zero. In [21] it was proved that if (ρ, τ) or $(\rho^{[1]}, \tau^{[1]})$ are prescribed, then there exist indeterminate moment problems with these (logarithmic) orders and types. In Ramis [22] the notion of logarithmic order and type appears for entire solutions to q-difference equations.

The main achievement of the present paper is that we present some conditions on the coefficients $(a_n), (b_n)$ of the three term recurrence relation (2), such that when these hold, then summability properties of the sequence $(P_n^2(z))$ and order properties of the moment problem are equivalent. Furthermore, the order as well as the logarithmic order of the moment problem can be calculated from the growth properties of the sequence (b_n).

These conditions are of two different types. There is a *regularity condition* that (b_n) is either log-convex eventually or log-concave eventually, cf. (27) or (28), and a *growth condition* (29).

The last condition is also necessary in the symmetric case $a_n = 0$ because of Carleman's condition.

We shall now give a more detailed survey of the content.