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Advances in Mathematics 250 (2014) 242–284

ADVANCES IN Mathematics

www.elsevier.com/locate/aim

Quantitative local and global a priori estimates for fractional nonlinear diffusion equations

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Received 7 December 2012; accepted 3 September 2013

Available online 14 October 2013

Communicated by Ovidiu Savin

Abstract

We establish quantitative estimates for solutions $u(t,x)$ to the fractional nonlinear diffusion equation, $\partial_t u + (-\Delta)^s(u^m) = 0$ in the whole range of exponents $m > 0$, $0 < s < 1$. The equation is posed in the whole space $x \in \mathbb{R}^d$. We first obtain weighted global integral estimates that allow to establish existence of solutions for classes of large data. In the core of the paper we obtain quantitative pointwise lower estimates of the positivity of the solutions, depending only on the norm of the initial data in a certain ball. The estimates take a different form in three exponent ranges: slow diffusion, good range of fast diffusion, and very fast diffusion. Finally, we show existence and uniqueness of initial traces.

MSC: 35B45; 35B65; 35K55; 35K65

Keywords: Nonlinear diffusion equation; Fractional Laplacian; Weighted global estimates; Existence for large data; Positivity estimates; Initial trace

Contents

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0001-8708/$ – see front matter
http://dx.doi.org/10.1016/j.aim.2013.09.018

1. Introduction

We consider the class of nonnegative weak solutions of the fractional diffusion equation

$$\partial_t u + (-\Delta)^s \big(u^m\big) = 0 \quad \text{in } (0, T) \times \mathbb{R}^d, \tag{1.1}$$

where $m > 0$, $0 < s < 1$, $d \geqslant 1$, and $T > 0$. The precise definition of the fractional Laplacian is given in Appendix A.1. We are mainly interested in values $m \neq 1$, since the linear case is rather well known. For $s = 1$ we recover the classical porous medium/fast diffusion equation (that will be shortened as PME/FDE respectively), whose theory is well known, cf. [38]. We will call the case $s = 1$ the standard diffusion case. We also assume that we are given initial data

$$u(0, x) = u_0(x), \tag{1.2}$$

where in principle $u_0 \in \mathrm{L}^1(\mathbb{R}^d)$ and $u_0 \geqslant 0$. However, larger classes of initial data are sometimes considered, like changing sign solutions or non-integrable data. The equation has recently attracted some attention in mathematical analysis. Such an interest has been motivated by its appearance as a model for anomalous diffusion in different applied contexts. For reader's convenience we have listed in Appendix B the most relevant sources to the applications that we know of.

We refer to [30,31] for the basic theory of existence and uniqueness of weak solutions for the Cauchy problem (1.1)–(1.2). These papers also describe the main results on L^q boundedness and C^α regularity, they show that nonnegative solutions are indeed positive everywhere, as well as some other basic properties of the nonlinear semigroup generated by the problem. Recently, the existence and properties of Barenblatt solutions for the Cauchy problem was established in [40]. Related literature is also mentioned in these papers.

The main purpose of the present paper is obtaining quantitative a priori estimates of a local type for the solutions of the problem. Such estimates were obtained for the standard PME by Aronson and Caffarelli [2] and by the authors for the standard FDE [9,11,10]. This is not always possible for the present model due to the nonlocal character of the diffusion operator, but then global estimates occur in weighted spaces. The use of suitable weight functions allows to prove