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The equivariant Euler characteristic of moduli spaces of curves

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Abstract

We derive a formula for the S_n-equivariant Euler characteristic of the moduli space $\mathcal{M}_{g,n}$ of genus g curves with n marked points.

Keywords: Moduli spaces; Equivariant Euler characteristic; Orbifold Euler characteristic

1. Introduction

Consider the moduli space $\mathcal{M}_{g,n}$ of algebraic curves of genus g with n marked points. The symmetric group S_n acts naturally on this space. Let V_λ denote the irreducible representation of S_n corresponding to a Young diagram λ, then one can decompose the cohomology of $\mathcal{M}_{g,n}$ into irreducible representations:

$$\mathrm{H}^i(\mathcal{M}_{g,n}) = \bigoplus_{\lambda} a_{i,\lambda} V_\lambda.$$

The S_n-equivariant Euler characteristic of $\mathcal{M}_{g,n}$ is defined by the formula

$$\chi^{S_n}(\mathcal{M}_{g,n}) = \sum_{i,\lambda} (-1)^i a_{i,\lambda} s_\lambda,$$

where s_λ denotes the Schur polynomial labeled by the diagram λ. We calculate these equivariant Euler characteristics for all $g \geqslant 2$ and n.

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Theorem 1.1. *The generating function for the S_n-equivariant Euler characteristics of $\mathcal{M}_{g,n}$ has the form*

$$\sum_{n=0}^{\infty} t^n \chi^{S_n}(\mathcal{M}_{g,n}) = \sum_{\underline{k}} c_{k_1,\ldots,k_r} \prod_{j=1}^{r} \left(1 + p_j t^j\right)^{k_j},$$

where p_j are power sums and the coefficients $c_{k_1,\ldots,k_r}$ are defined by Eq. (6).

Consider the moduli space $\mathcal{M}_g(k_1,\ldots,k_r)$ of pairs (C,τ) where C is a genus g curve and τ is an automorphism of C such that for all i the Euler characteristic of the set of points in C having the orbit of length i under the action of τ equals ik_i. Note that the coefficient $c_{k_1,\ldots,k_r}$ also can be defined as the orbifold Euler characteristic of $\mathcal{M}_g(k_1,\ldots,k_r)$.

This moduli space can be defined for any tuple of integers $(k_1,\ldots,k_r)$ of arbitrary size r, but we prove that (for a fixed genus g) it is non-empty only for a finite number of such tuples. In particular, r cannot exceed $4g+2$.

Corollary 1.2. *The generating function $\sum_{n=0}^{\infty} t^n \chi^{S_n}(\mathcal{M}_{g,n})$ is a rational function in t. Furthermore, for any n $\chi^{S_n}(\mathcal{M}_{g,n}) \in \mathbb{Z}[p_1,\ldots,p_{4g+2}]$.*

Theorem 1.1 can be compared with the computations of [4,5,8,10] in genus 2 and with the computations of [1,2,9,17] in genus 3. A similar generating function for the moduli spaces of hyperelliptic curves was previously obtained in [11].

The paper is organized as follows. In Section 2 we consider a complex quasi-projective variety X with an action of a finite group G. Theorem 2.5 provides a formula for the S_n-equivariant Euler characteristic of quotients $F(X,n)/G$, where $F(X,n)$ is a configuration space of n labeled distinct points on X. This theorem was previously proved in [10] using the results of Getzler [6,7] concerning Adams operations over the equivariant motivic rings (see also [12]). The alternative proof presented here uses only the basic properties of Euler characteristic and seems to be more geometric. It also makes the proof of the main result self-contained.

In Section 3 we apply this theorem to the universal family over $\mathcal{M}_g$, the moduli space of genus g curves. This allows us to prove in Theorem 3.3 that the coefficients $c_{k_1,\ldots,k_r}$ are equal to the orbifold Euler characteristic of $\mathcal{M}_g(k_1,\ldots,k_r)$. These Euler characteristics are then computed in Theorem 3.8 using the results of Harer and Zagier.

2. Equivariant Euler characteristics

Let X be a complex quasi-projective variety with an action of a finite group G. Let us denote by $F(X,n)$ the configuration space of ordered n-tuples of distinct points on X. For each n, the action of the group G on X can be naturally extended to the action of G on $F(X,n)$, commuting with the natural action of S_n.

In the computations below we will use the additivity and multiplicativity of the Euler characteristic, as well as the Fubini formula for the integration with respect to the Euler characteristic ([15,18], see also [16]).

Lemma 2.1. *The following equation holds*: $\sum_{n=0}^{\infty} \frac{t^n}{n!} \chi(F(X,n)) = (1+t)^{\chi(X)}$.

Proof. The map $\pi_n : F(X,n) \to F(X,n-1)$, which forgets the last point in the n-tuple, has fibers isomorphic to X without $n-1$ points. Therefore $\chi(F(X,n)) = (\chi(X)-n+1) \cdot \chi(F(X,n-1))$, and $\chi(F(X,n)) = \chi(X) \cdot (\chi(X)-1) \cdot \cdots \cdot (\chi(X)-n+1)$. $\square$