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Local cohomology with support in ideals of maximal minors and sub-maximal Pfaffians

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Abstract

We compute the GL-equivariant description of the local cohomology modules with support in the ideal of maximal minors of a generic matrix, as well as of those with support in the ideal of $2n \times 2n$ Pfaffians of a $(2n+1) \times (2n+1)$ generic skew-symmetric matrix. As an application, we characterize the Cohen–Macaulay modules of covariants for the action of the special linear group $\mathrm{SL}(G)$ on $G^{\oplus m}$. The main tool we develop is a method for computing certain Ext modules based on the geometric technique for computing syzygies and on Matlis duality.
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1. Introduction

In this paper we present the GL-equivariant description of the local cohomology modules of a polynomial ring S with support in an ideal I (denoted $H_I^j(S)$ for $j \geqslant 0$), in two cases of interest (throughout the paper, $\mathbb{K}$ will denote a field of characteristic zero):

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- S is the ring of polynomial functions on the vector space of $m \times n$ matrices with entries in $\mathbb{K}$, and I is the ideal of S generated by the polynomial functions that compute the $n \times n$ minors.
- S is the ring of polynomial functions on the vector space of $(2n+1) \times (2n+1)$ skew-symmetric matrices with entries in $\mathbb{K}$, and I is the ideal generated by the polynomial functions that compute the $2n \times 2n$ Pfaffians.

One of the motivations behind our investigation is trying to understand the Cohen–Macaulayness of modules of covariants. This problem has a long history, originating in the work of Stanley [16] on solution sets of linear Diophantine equations (see [20] for a survey, and also [3,18,19,21]). When H is a reductive group and W a finite dimensional H-representation, a celebrated theorem of Hochster and Roberts [11] asserts that the ring of invariants S^H, with respect to the natural action of H on the polynomial ring $S = \mathrm{Sym}(W)$, is Cohen–Macaulay. If U is another finite dimensional H-representation, the associated *module of covariants* is defined as $(S \otimes U)^H$, and is a finitely generated S^H-module. In general it is quite rare that $(S \otimes U)^H$ is Cohen–Macaulay, and our first result illustrates this in a special situation.

Theorem on Covariants of the Special Linear Group. *(Theorem 4.6) Consider a finite dimensional $\mathbb{K}$-vector space G of dimension n, an integer $m > n$, and let $H = \mathrm{SL}(G)$ be the special linear group, $W = G^{\oplus m}$, and $S = \mathrm{Sym}(W)$. If $U = S_\mu G$ is the irreducible H-representation associated to the partition $\mu = (\mu_1 \geqslant \mu_2 \geqslant \cdots \geqslant \mu_n = 0)$ then $(S \otimes U)^H$ is Cohen–Macaulay if and only if $\mu_s - \mu_{s+1} < m - n$ for all $s = 1, \ldots, n-1$.*

In the case when $m = n+1$, the Theorem on Covariants asserts that the only Cohen–Macaulay modules of covariants are direct sums of copies of S^H, which is remarked at the end of [3]. The case $n = 3$ of the theorem is explained in [21], while the case $n = 2$ for an arbitrary H-representation W is treated in [19]. We restrict to $m > n$ in the statement of the theorem to avoid trivial cases: if $m < n$ then $S^H = \mathbb{K}$, while for $m = n$, $S^H = \mathbb{K}[\det]$ is a polynomial ring in one variable det, corresponding to the determinant of the generic $n \times n$ matrix; in both cases, all the modules of covariants are Cohen–Macaulay.

We will prove the Theorem on Covariants by computing explicitly the local cohomology modules $H_I^j(S)$, where I is the ideal generated by the maximal minors of a generic $m \times n$ matrix, and using the relationship between the local cohomology of the module $(S \otimes U)^H$ and the invariants $(H_I^j(S) \otimes U)^H$ [17, Lemma 4.1]. The indices j for which $H_I^j(S) \neq 0$ have been previously determined by the third author, as well as the description of the top non-vanishing local cohomology module $H_I^{n\cdot(m-n)+1}(S)$ as the injective hull of the residue field: see [24] for details, including some history behind the problem and its positive characteristic analogue. Writing $W = G^{\oplus m} = F \otimes G$ for some m-dimensional vector space F, we note that the ideal I (and hence the local cohomology modules $H_I^j(S)$) is preserved by the action of the product $\mathrm{GL}(F) \times \mathrm{GL}(G)$ of general linear groups. Our explicit description of the modules $H_I^j(S)$ exhibits their decomposition as a direct sum of irreducible representations of this group.

Theorem on Maximal Minors. *(Theorem 4.5) Consider $\mathbb{K}$-vector spaces F and G of dimensions m and n respectively, with $m > n$, and integers $r \in \mathbb{Z}$, $j \geqslant 0$. For $1 \leqslant s \leqslant n$ and $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n$ a dominant weight (i.e. $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n$), let*

$$\lambda(s) = \big(\lambda_1, \ldots, \lambda_{n-s}, \underbrace{-s, \ldots, -s}_{m-n}, \lambda_{n-s+1} + (m-n), \ldots, \lambda_n + (m-n)\big).$$

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