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Relations between the minors of a generic matrix

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To David Eisenbud on his 65th birthday

Abstract

It is well-known that the Plücker relations generate the ideal of relations of the maximal minors of a generic $m \times n$ matrix. In this paper we discuss the relations of t-minors for $t < \min(m, n)$. We will exhibit minimal relations in degrees 2 (non-Plücker in general) and 3, and give some evidence for our conjecture that we have found the generating system of the ideal of relations. The approach is through the representation theory of the general linear group.

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0. Introduction

In algebra, algebraic geometry and representation theory the polynomial relations between the minors of a matrix are interesting objects for many reasons. Surprisingly they are still unknown in almost all cases. While it is a classical theorem that the Plücker relations (of maximal minors of a generic matrix) generate the defining ideal of the Grassmannian, only a few other cases have been treated, for example, the principal minors of a (symmetric) matrix; see Holtz and Sturmfels [14],

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Lin and Sturmfels [16] and Oeding [18]. For arbitrary t, the relations between the t-minors of a generic matrix are certainly not understood, and in this paper we try to investigate them.

We refer the reader to Fulton and Harris [13], Procesi [19], and Weyman [20] for background in representation theory, to Bruns and Vetter [8] for the theory of determinantal rings, and to [2–5] for structural results of algebras generated by minors.

Let us consider the matrix

$$X = \begin{pmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & x_{22} & x_{23} & x_{24} \end{pmatrix}$$

where the x_{ij}'s are indeterminates over a field $\Bbbk$. With $[ij] = x_{1i}x_{2j} - x_{1j}x_{2i}$, one has

$$[12][34] - [13][24] + [14][23] = 0.$$

This is the Plücker relation, and it is the only minimal relation in the sense that it generates the ideal of relations. In fact, the case $t = \min\{m, n\}$ is well understood in general, even if anything but trivial: if $t = \min\{m, n\}$ the Plücker relations generate the ideal of relations between the t-minors of X. In particular, there are only quadratic minimal relations. Similarly, other classical algebras generated by minors, like the coordinate ring of the flag variety, are defined by quadrics; for instance see [17, Chap. 14].

This changes already for 2-minors of a 3×4-matrix. To identify a minor we have now to specify row and column indices. Denote by $[ij|pq]$ the minor of X with row indices i, j and column indices p, q. Of course, the Plücker relations are still present, but they are no more sufficient. Cubics appear among the minimal relations, for example

$$\det \begin{pmatrix} [12|12] & [12|13] & [12|14] \\ [13|12] & [13|13] & [13|14] \\ [23|12] & [23|13] & [23|14] \end{pmatrix} = 0; \tag{0.1}$$

see [2].

One reason why the case of maximal minors is easier than the general case emerges from a representation-theoretic point of view. Let $\Bbbk$ be a field of characteristic 0, A_t denote the subalgebra of the polynomial ring $\Bbbk[X] = \Bbbk[x_{ij}]$ generated by the t-minors of X. When $t = m \le n$ the ring A_t is the coordinate ring of the Grassmannian $G(m, n)$ of all m-dimensional subspaces of a vector space W of dimension n. In the general case, A_t is the coordinate ring of the Zariski closure of the image of the following morphism of affine spaces:

$$\Lambda_t : \mathrm{Hom}_{\Bbbk}(W, V) \to \mathrm{Hom}_{\Bbbk}\left(\bigwedge^t W, \bigwedge^t V\right), \qquad \Lambda_t(\phi) = \wedge^t \phi,$$

where V is a vector space of dimension m. Notice that the group $G = \mathrm{GL}(V) \times \mathrm{GL}(W)$ acts on each graded component $(A_t)_d$ of A_t. If $t = \min\{m, n\}$, then each $(A_t)_d$ is actually an *irreducible* G-representation. This is far from being true in the general case, and this complicates the situation tremendously.

In this paper we will exhibit quadratic and cubic minimal relations between t-minors, that naturally appear in an $m \times n$-matrix for $t \ge 2$. The action of G on A_t induces a G-action also on the ideal of relations J_t. Therefore it suffices to describe the highest weight vectors of the G-irreducible subrepresentations of J_t.

Each relation f between minors gives rise to a mirror relation denoted by f', namely the one obtained by switching columns and rows.