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The Liouville theorem under second order differentiability assumption

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Abstract

Iwaniec and Martin proved that in even dimensions $n \geq 3$, $W^{1,n/2}_{\mathrm{loc}}$ conformal mappings are Möbius transformations and they conjectured that it should also be true in odd dimensions. We prove this theorem for a conformal map $f \in W^{1,1}_{\mathrm{loc}}$ in dimension $n \geq 3$ under one additional assumption that the norm of the first order derivative $|Df|$ satisfies $|Df|^p \in W^{1,2}_{\mathrm{loc}}$ for $p \geq (n-2)/4$. This is optimal in the sense that if $|Df|^p \in W^{1,2}_{\mathrm{loc}}$ for $p < (n-2)/4$, it may not be a Möbius transform. This result shows the necessity of the Sobolev exponent in the Iwaniec–Martin conjecture. Meanwhile, we show that the Iwaniec–Martin conjecture can be reduced to a conjecture about the Caccioppoli type estimate.

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1. Introduction

1.1. History

A diffeomorphism of class C^1 is said to be conformal if it preserves the angle between any two curves at each point of its domain. In complex analysis, we know that every holomorphic function on an open set with non-vanishing derivative is a conformal map. Thus the plane is rich in conformal maps. Moreover, the Riemann mapping theorem asserts in the plane any simply connected domain that is not the entire plane is conformally equivalent to the disk.

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However, in dimension $n \geq 3$, the only conformal maps of class C^1 are Möbius transformations, that is, mappings generated by translations, rotations, dilatations, reflections, and inversions in spheres. For C^3 conformal diffeomorphisms in $\mathbb{R}^3$, Liouville [14] established this result in 1850. This is a strong rigidity theorem. In particular, it implies the only domains conformally equivalent to the unit ball are balls or half spaces, a sharp contrast to the Riemann mapping theorem. In fact the Riemann mapping theorem was announced by Riemann just a year after the Liouville result. Capelli [4] in 1886 extended the Liouville theorem to all higher dimensions for maps also of class C^3. Another well-known proof is the one given by Nevanlinna [16,6] in 1960 under a C^4 smoothness assumption using elementary tools of analysis.

Proving the Liouville theorem under weaker regularity assumptions, however, turned out to be difficult. In 1947, Hartman [8] proved the theorem for C^2 conformal diffeomorphisms and later in 1958 [9] for the C^1 case. Another proof of the Liouville theorem for C^2 conformal diffeomorphisms was given by Sarvas [18] in 1978.

Since the Liouville theorem plays a crucial role in the theory of quasiconformal mappings and in the non-convex calculus of variations, there is a need for proving the theorem under still weaker assumptions. The right setting turns out to be the setting of Sobolev spaces $W^{1,p}$, i.e., the space of all functions in the L^p space whose distributional derivatives are represented by L^p functions. The case of 1-quasiconformal mappings, that is, conformal homeomorphisms in the Sobolev space $W^{1,n}$, was treated by Gehring [7] in 1962 and later by Reshetnyak [17] without the homeomorphism assumption. Both authors refer to difficult results in nonlinear PDEs, geometry, and the theory of quasiconformal mappings. An elementary, but rather involved proof of Reshetnyak's result was given by Bojarski and Iwaniec [3] in 1980, also see [13,12]. More recent developments have arisen from the work of Iwaniec and Martin [11], where they proved the Liouville theorem for $W^{1,n/2}$ conformal maps in Euclidean spaces of even dimensions n. Meanwhile, they gave a counter-example (Example 5.2) showing that in all dimensions greater than or equal to three, conformal maps in the space $W^{1,p}$ for $p < n/2$ may not be Möbius. Whether the Liouville theorem holds for $W^{1,n/2}$ conformal maps for all dimensions $n \geq 3$ remains a challenging open problem.

Let us turn our attention to some versions of the Liouville theorem in the Sobolev spaces setting.

Let Ω be a domain in $\mathbb{R}^n$. A function $f \in W^{1,1}_{\text{loc}}(\Omega, \mathbb{R}^n)$ is *conformal* or a *weak solution* to the *Cauchy–Riemann system* if,

$$(Df(x))^T\, Df(x) = |J(x, f)|^{2/n} \cdot \mathrm{I} \quad \text{and,} \tag{1.1}$$

$$J(x, f) \geq 0 \quad \text{a.e.} \quad \text{or} \quad J(x, f) \leq 0 \quad \text{a.e.,} \tag{1.2}$$

where Df is the weak differential of f, i.e., the matrix of weak partial derivatives of f, and $J(x, f) = \det Df(x)$. We call f with $J(x, f) \geq 0$ a.e. *sense preserving* and $J(x, f) \leq 0$ a.e. *sense reversing*. Note that in dimensions $n \geq 3$, this is an over-determined system. That is why the situation is more rigid than in the plane.

A *Möbius transform* is a composition of translations, dilatations, rotations, reflections, and inversions with respect to spheres. More precisely, it has the following form

$$f(x) = b + \frac{r^2 A(x - a)}{|x - a|^\alpha} \tag{1.3}$$

where $b \in \mathbb{R}^n$, $r \in \mathbb{R} \setminus \{0\}$, $a \in \mathbb{R}^n \setminus \Omega$, A an orthogonal matrix, and α is either 0 or 2.