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Heegaard–Floer homology of broken fibrations over the circle

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Abstract

We extend Perutz's Lagrangian matching invariants to 3-manifolds which are not necessarily fibered using the technology of holomorphic quilts. We prove an isomorphism of these invariants with Ozsváth–Szabó's Heegaard–Floer invariants for certain extremal spin^c structures. As applications, we give new calculations of Heegaard–Floer homology of certain classes of 3-manifolds, and a characterization of Juhász's sutured Floer homology.

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1. Introduction

In this paper, we study two seemingly different Floer theoretical invariants of three- and four-manifolds. These are Perutz's Lagrangian matching invariants and Ozsváth and Szabó's Heegaard–Floer theoretical invariants. The main result of this paper is an isomorphism between the 3-manifold invariants of these theories for certain spin^c structures, namely *quilted Floer homology* and *Heegaard–Floer homology*. We also outline how the techniques here can be generalized to obtain an identification of 4-manifold invariants.

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Before giving a review of both of the above mentioned theories, we give the definition of a broken fibration over S^1, which will be an important part of the topological setting that we will be working with.

Definition 1. A map $f : Y \to S^1$ from a closed oriented smooth 3-manifold Y to S^1 is called a broken fibration if f is a circle-valued Morse function with all of the critical points having index 1 or 2.

The terminology is inspired from the terminology of broken Lefschetz fibrations on 4-manifolds, to which we will return later in this paper in Section 5. We remark that a 3-manifold admits a broken fibration if and only if $b_1(Y) > 0$, and if it admits one, it admits a broken fibration with connected fibers.

We will restrict ourselves to broken fibrations with connected fibers and we will denote by $\Sigma_{\max}$ and $\Sigma_{\min}$ two fibers with maximal and minimal genus respectively. We denote by $\mathcal{S}(Y|\Sigma_{\min})$, the spin^c structures $\mathfrak{s}$ on Y such that $\langle c_1(\mathfrak{s}), [\Sigma_{\min}]\rangle = \chi(\Sigma_{\min})$ (those spin^c structures which satisfy the adjunction *equality* with respect to the fiber with minimal genus).

Definition 2. The universal Novikov ring Λ over $\mathbb{Z}_2$ is the ring of formal power series $\Lambda = \Sigma_{r\in\mathbb{R}}\, a_r t^r$ with $a_r \in \mathbb{Z}_2$ such that $\#\{r|a_r \neq 0, r < N\} < \infty$ for any $N \in \mathbb{R}$.

We first give a definition of a new invariant $QFH'(Y, f, \mathfrak{s}; \Lambda)$ for all spin^c structures in $\mathcal{S}(Y|\Sigma_{\min})$ and prove an isomorphism between this variant of quilted Floer homology of a broken fibration $f : Y \to S^1$ (with coefficients in the universal Novikov ring) and the Heegaard–Floer homology of Y perturbed by a closed 2-form η that pairs positively with the fibers of f.

Theorem 3. *$QFH'(Y, f, \mathfrak{s}; \Lambda) \simeq HF^{\pm}(Y, \eta, \mathfrak{s})$ for $\mathfrak{s} \in \mathcal{S}(Y|\Sigma_{\min})$.*

When $g(\Sigma_{\min})$ is at least 2, the coefficients can be taken to be in $\mathbb{Z}_2$ (in this case admissibility of our diagrams are automatic, therefore we do not need to use perturbations).

Corollary 4. *Suppose that $g(\Sigma_{\min}) > 1$. Then for $\mathfrak{s} \in \mathcal{S}(Y|\Sigma_{\min})$ we have*

$$QFH'(Y, f, \mathfrak{s}; \mathbb{Z}_2) \simeq HF^+(Y, \mathfrak{s}; \mathbb{Z}_2).$$

As corollaries of this result, we give new calculations of Heegaard Floer homology groups for certain manifolds for which $QFH'(Y, f, \mathfrak{s})$ is easy to calculate. We give several such calculations among which the following is particularly interesting.

Corollary 5. *Suppose f has only two critical points, and let $\alpha, \beta \subset \Sigma_{\max}$ be the vanishing cycles of these critical points. Then, $\oplus_{\mathfrak{s}\in\mathcal{S}(Y|\Sigma_{\min})} HF^+(Y, \eta, \mathfrak{s})$ is free of rank $\iota(\alpha, \beta)$, the geometric intersection number between α and β. Furthermore, if $g(\Sigma_{\min}) > 1$ then the result holds over $\mathbb{Z}_2$, i.e.*

$$\oplus_{\mathfrak{s}\in\mathcal{S}(Y|\Sigma_{\min})} HF^+(Y, \mathfrak{s}) = \mathbb{Z}_2^{\iota(\alpha,\beta)}.$$

The second main theorem proves that the invariants $QFH'(Y, f, \mathfrak{s}; \Lambda)$ that we defined are isomorphic to the quilted Floer homology groups coming from Perutz's theory of Lagrangian matching invariants. Unlike $QFH'(Y, f, \mathfrak{s}; \Lambda)$, for technical reasons these latter invariants are only defined in the case $g(\Sigma_{\max}) < 2g(\Sigma_{\min})$. Thus, we have the following theorem.