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Spectra of combinatorial Laplace operators on simplicial complexes

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Abstract

We first develop a general framework for Laplace operators defined in terms of the combinatorial structure of a simplicial complex. This includes, among others, the graph Laplacian, the combinatorial Laplacian on simplicial complexes, the weighted Laplacian, and the normalized graph Laplacian. This framework then allows us to define the normalized Laplace operator Δ_i^{up} on simplicial complexes which we then systematically investigate. We study the effects of a wedge sum, a join and a duplication of a motif on the spectrum of the normalized Laplace operator and identify some of the combinatorial features of a simplicial complex that are encoded in its spectrum.

Keywords: Normalized graph Laplacian; Combinatorial Laplacian; Hypergraph Laplacian; Graph Laplacian; Simplicial complex

1. Introduction

The study of graph Laplacians has a long and prolific history. It first appeared in a paper by Kirchhoff [25], where he analysed electrical networks and stated the celebrated matrix tree

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theorem. The Laplace operator L of [25] operates on a real valued function f on the vertices of a graph as

$$Lf(v_i) = \deg v_i f(v_i) - \sum_{v_i \sim v_j} f(v_j). \tag{1.1}$$

In spite of its rather early beginnings this topic did not gain much attention among scientists until the early 1970s and the work of Fiedler [14], and his results on correlation among the smallest non-zero eigenvalue and the connectivity of a graph. Before Fiedler drew attention to the graph Laplacian, graphs were usually characterized by means of the spectrum of its adjacency matrix, but in the wake of [14], there has been a number of papers (e.g. [19]) arguing in favour of the graph Laplacian and its spectrum. For good survey articles on the graph Laplacian the reader is referred to [28] or [30].

In a different tradition, the graph Laplacian was generalized to simplicial complexes by Eckmann [13], who formulated and proved the discrete version of the Hodge theorem; this can be formulated as

$$\ker(\delta_i^* \delta_i + \delta_{i-1} \delta_{i-1}^*) \cong \tilde{H}^i(K, \mathbb{R})$$

where

$$L_i = \delta_i^* \delta_i + \delta_{i-1} \delta_{i-1}^*$$

is the higher order combinatorial Laplacian. Many subsequent papers then studied properties of the *higher order combinatorial Laplacian* (see [10,15,11]), building upon properties of the graph Laplacian. In particular, this operator has been employed extensively in investigating the features of networks related to dynamics and coverings (see [31,33]). More recently, in [23], the combinatorial Laplacian is systematically studied in a context of a discrete exterior calculus.

While the graph Laplacian introduced by Kirchhoff naturally appears in his work on electrical flows, for other processes on graphs, like random walks or diffusion, a different operator appears. This was first investigated almost a century after Kirchhoff's work by Bottema [5] who studied a transition probability operator on graphs that is equivalent to the following version of the graph Laplace operator:

$$\Delta f(v_i) = f(v_i) - \frac{1}{\deg v_i} \sum_{v_i \sim v_j} f(v_j). \tag{1.2}$$

It took, however, almost another one hundred years until a significant advance in the study of this operator Δ, which got to be known by the name *normalized graph Laplacian* to distinguish it from the graph Laplacian L and to emphasize the fact that its eigenvalues are in the interval $[0, 2]$. In contrast to L, Δ is well suited for problems related to random walks on graphs and graph expanders. For a good introduction to this topic the reader is invited to consult [6] or [18].

The main goals of this paper are a systematic framework that can be used as a starting point for a study of any of the above mentioned versions of the Laplace operator, and the definition and investigation of the normalized Laplacian on simplicial complexes. The latter is based on the simple observation that the form of the combinatorial Laplacian is tightly connected to the choice of the scalar product on the coboundary vector spaces. On the other hand, the scalar products can be viewed in terms of weight functions. Thus, by controlling the weights, we control the range