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A sharp growth condition for a fast escaping spider's web

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Abstract

We show that the fast escaping set $A(f)$ of a transcendental entire function f has a structure known as a spider's web whenever the maximum modulus of f grows below a certain rate. The proof uses a new local version of the $\cos \pi\rho$ theorem, based on a comparatively unknown result of Beurling. We also give examples of entire functions for which the fast escaping set is not a spider's web which show that this growth rate is sharp. These are the first examples for which the escaping set has a spider's web structure but the fast escaping set does not. Our results give new insight into possible approaches to proving a conjecture of Baker, and also a conjecture of Eremenko.

MSC: primary 37F10; secondary 30D05

Keywords: Fast escaping set; Spiders' webs; $\cos \pi\rho$ theorem

1. Introduction

Let $f : \mathbb{C} \to \mathbb{C}$ be a transcendental entire function and denote by f^n, $n = 0, 1, 2, \ldots$, the nth iterate of f. The *Fatou set* $F(f)$ is the set of points $z \in \mathbb{C}$ such that $(f^n)_{n\in\mathbb{N}}$ forms a normal family in some neighborhood of z. The complement of $F(f)$ is called the *Julia set* $J(f)$ of f. An introduction to the properties of these sets can be found in [3].

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In recent years, the escaping set defined by

$$I(f) = \{z : f^n(z) \to \infty \text{ as } n \to \infty\}$$

has come to play an increasingly significant role in the study of the iteration of transcendental entire functions with much of the research being motivated by a conjecture of Eremenko [5] that all the components of the escaping set are unbounded. For partial results on this conjecture see, for example, [9,15].

The most general result on Eremenko's conjecture was obtained in [10] where it was proved that the escaping set always has at least one unbounded component. This result was proved by considering the fast escaping set $A(f) = \bigcup_{n\in\mathbb{N}} f^{-n}(A_R(f))$, where

$$A_R(f) = \{z : |f^n(z)| \geq M^n(R, f), \text{ for } n \in \mathbb{N}\}.$$

Here

$$M(r) = M(r, f) = \max_{|z|=r} |f(z)|,$$

$M^n(r, f)$ denotes the nth iterate of M with respect to r, and $R > 0$ is chosen so that $M(r, f) > r$ for $r \geq R$. The set $A(f)$ has many nice properties including the fact that all its components are unbounded—these properties are described in detail in [12].

There are many classes of transcendental entire functions for which the fast escaping set has the structure of a spider's web—see [12,8,16]. We say that a set E has this structure if E is connected and there exists a sequence of bounded simply connected domains G_n such that

$$\partial G_n \subset E, \qquad G_n \subset G_{n+1}, \quad \text{for } n \in \mathbb{N}, \quad \text{and} \quad \bigcup_{n\in\mathbb{N}} G_n = \mathbb{C}.$$

As shown in [12], if $A_R(f)$ has this structure then so do both $A(f)$ and $I(f)$, and hence Eremenko's conjecture is satisfied. Also, the domains G_n can be chosen so that $\partial G_n \subset A_R(f) \cap J(f)$ and so f has no unbounded Fatou components. This gives a surprising link between Eremenko's conjecture and a conjecture of Baker [1] that all the components of the Fatou set are bounded if f is a transcendental entire function of order less than $1/2$. Recall that the *order* of a transcendental entire function f is defined to be

$$\rho = \limsup_{r\to\infty} \frac{\log\log M(r)}{\log r}.$$

For background and recent results on Baker's conjecture, see [6,7,11,13]. It was shown in [11] (see also [12]) that the techniques used to prove all earlier partial results on Baker's conjecture can in fact be used to prove the stronger result that $A_R(f)$ is a spider's web.

In this paper we show the limitation of these techniques, and demonstrate that they cannot even be used to prove Baker's conjecture for all functions of order zero. To do this we give a *sharp* condition on the growth of the maximum modulus that is sufficient to imply that $A_R(f)$ is a spider's web and hence that Baker's conjecture and Eremenko's conjecture are both satisfied. More precisely, we prove the following sufficient condition.

Theorem 1.1. *Let f be a transcendental entire function and let $R > 0$ be such that $M(r, f) > r$ for $r \geq R$. Let*

$$R_n = M^n(R) \quad \textit{and} \quad \varepsilon_n = \max_{R_n \leq r \leq R_{n+1}} \frac{\log\log M(r)}{\log r}.$$