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On weak product recurrence and synchronization of return times

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Abstract

The paper is devoted to a study of product recurrence. First, we prove that notions of $\mathscr{F}_{ps}$-PR and $\mathscr{F}_{pubd}$-PR are exactly the same as product recurrence, completing that way results of [P. Dong, S. Shao, X. Ye, Product recurrent properties, disjointness and weak disjointness, Israel J. Math. 188 (1) (2012) 463–507], and consequently, extending the characterization of return times of distal points which originated from the works of Furstenberg. We also study the structure of the set of return times of weakly mixing sets. As a consequence, we obtain new sufficient conditions for $\mathscr{F}_s$-PR and also find a short proof that weakly mixing systems are disjoint with all minimal distal systems (in particular, our proof does not involve Furstenberg's structure theorem of minimal distal systems).

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1. Introduction

There are two important characterizations related to distality, both observed by Furstenberg more than 30 years ago. If a point is distal, then it is recurrent in pairs with any recurrent point

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in any dynamical system [17] and if a minimal dynamical system is distal then it is disjoint with any weakly mixing system [16]. It is worth emphasizing that later, a full characterization of flows disjoint with all distal flows was provided by Petersen in [31]. While proofs of both the above mentioned characterizations are not that long, their proofs highly rely on other, even more important and highly nontrivial results. Namely, the first of them uses Hindman's theorem on finite partitions of IP-sets, while the second can be obtained as a consequence of an algebraic characterization of distal flows, proved by Furstenberg in [15].

The above characterizations on synchronization of return times of a point with return times of distal points gave motivation to two directions of research. The first of them asks about synchronization of return times between different types of recurrent points and the second asks about disjointness between specified classes of systems. Both these questions lead to partial classifications and hard open problems [11]. It is also worth emphasizing that studies on the above topics were very influential and important for topics which at first sight do not seem to be related very much. For example, Blanchard characterized in [8] systems disjoint with flows with zero entropy giving in that way a good motivation for introducing entropy pairs and systems with uniform positive entropy (u.p.e.) which are two fundamental notions in local entropy theory which were used later for deep and insightful investigations on topological entropy (cf. [19] for more on local entropy theory). In general, results on synchronization of trajectories can be of wide use (e.g. they can help to simplify some arguments in proofs etc.). For example, analysis of return or transfer times of points lead to a simple proof that a distal point is always minimal or that point minimal or recurrent for a dynamical system (X, f) is, respectively, minimal and recurrent for (X, f^n) for every n, etc.

If in place of recurrence in pairs with any recurrent point we demand recurrence in pairs with points in a smaller class of dynamical systems, it can lead to a wider class of points than the class of all distal points. For example, Auslander and Furstenberg in [5] asked about points which are recurrent in pairs with any minimal point. While there is no known full characterization of points with this property, it was proved in [20] that the class of such points is much larger than distal points, in particular it contains many points which are not minimal. Other sufficient conditions for this kind of product recurrence were provided in [11,28]. Moreover, [11] defines product recurrence in terms of Furstenberg families (i.e. upward hereditary sets of subsets of $\mathbb{N}$), which is a nice tool for a better classification of product recurrence. It is worth emphasizing that the concepts of [11] are not artificial, since it is possible almost immediately to relate these new types of product recurrence with some older results on disjointness.

The present paper completes some previous studies on recurrence and product recurrence from [11,28]. First, we prove that notions of $\mathscr{F}_{ps}$-PR and $\mathscr{F}_{pubd}$-PR are exactly the same as product recurrence, that is, if return times of a point can be synchronized with points returning with a piecewise syndetic set of times, then it can be synchronized with any recurrent point. This provides another condition to the list of conditions equivalent to distality as first provided by Furstenberg in [17, Theorem 9.11], and next extended by many authors (e.g. see [11]). Next, we analyze synchronizing properties of points in weakly mixing sets, which allow us to show that any weakly mixing set with dense distal points contains a residual subset of $\mathscr{F}_s$-PR points which are not distal (this is a question left open in [28]) and also to prove Furstenberg's result on disjointness between weakly mixing systems and minimal distal systems without referring to Furstenberg's structure theorem of distal flows. This is especially nice, since now both results of Furstenberg mentioned in the first paragraph of this introduction can be obtained using only Hindman's theorem plus some topological arguments. This even more bonds these two results together.