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Infinitesimal deformations of nodal stable curves

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Abstract

An analytic approach and description are presented for the moduli cotangent sheaf for suitable stable curve families including noded fibers. For sections of the square of the relative dualizing sheaf, the residue map at a node gives rise to an exact sequence. The residue kernel defines the vanishing residue subsheaf. For suitable stable curve families, the direct image sheaf on the base is locally free and the sequence of direct images is exact. Recent work of Hubbard–Koch and a formal argument provide that the direct image sheaf is naturally identified with the moduli cotangent sheaf. The result generalizes the role of holomorphic quadratic differentials as cotangents for smooth curve families. Formulas are developed for the pairing of an infinitesimal opening of a node and a section of the direct image sheaf. Applications include an analytic description of the conormal sheaf for the locus of noded stable curves and a formula comparing infinitesimal openings of a node. The moduli action of the automorphism group of a stable curve is described. An example of plumbing an Abelian differential and the corresponding period variation is presented.

MSC: primary 14H15; 32G15

Keywords: Nodal stable curves; Noded Riemann surfaces; Plumbing construction; Holomorphic quadratic differentials; Moduli cotangent bundle

1. Introduction

A torus T with fundamental group marking is uniformized by the complex plane $\mathbb{C}$ with variable z and a lattice generated by 1 and τ, τ in the upper half plane $\mathbb{H}$. The change of marking equivalence relation for tori is given by the action of the modular group $SL(2;\mathbb{Z})$ on $\mathbb{H}$. The Grötzsch and Rauch variational formulas provide that the differential of the moduli parameter τ is represented by the quadratic differential $-2i dz^2 \in H^0(T_\tau, \mathcal{O}(K^2))$ for K the canonical bundle. With the analogy to higher genus moduli in mind, consider $\mathbb{H}$ as the Teichmüller space, $SL(2;\mathbb{Z})$

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as the mapping class group and the quotient as the moduli space. The compactification of the quotient $\mathbb{H}/SL(2;\mathbb{Z})$ is given by introducing the coordinate $\mathbf{t} = e^{2\pi i\tau}$ for a neighborhood of infinity. The differential of the nonzero moduli parameter $\mathbf{t}$ is $4\pi\mathbf{t}dz^2 \in H^0(T_\tau, \mathcal{O}(K^2))$, which formally vanishes at $\tau = i\infty$ or equivalently $d\mathbf{t}/\mathbf{t}$ is represented by the quadratic differential $4\pi dz^2$. In particular, a quadratic differential on the limiting cylinder models the logarithmic derivative of the moduli parameter at infinity. Equivalently, the moduli cotangent $4\pi\mathbf{t}dz^2$ at infinity includes a $\mathbf{t}$ factor. We consider the corresponding results for families of stable curves.

The infinitesimal deformation space of a pair (R, q), a compact Riemann surface and a distinguished point, is the cohomology group $H^1(R, \mathcal{O}(K^{-1}q^{-1}))$ for the canonical bundle K and the inverse of the point bundle. By Kodaira–Serre duality, the dual infinitesimal deformation space is $H^0(R, \mathcal{O}(K^2q))$, the space of holomorphic quadratic differentials with possible simple poles at q. Quadratic differentials with simple poles at distinguished points give the moduli cotangent space. Our goal is to generalize this result for stable curves and describe the moduli cotangent space as a coherent analytic sheaf on the family.

Our discussion begins with the local geometry of the model case, the family of hyperbolas $zw = t$ in $\mathbb{C}^3$. Consider for c, c' positive, the singular fibration of $V = \{|z| < c, |w| < c'\} \subset \mathbb{C}^2$ over $D = \{|t| < cc'\}$ with the projection $\pi(z, w) = zw$. The family $\pi : V \to D$ is the local model for the formation and deformation of a node. The general fiber is an annulus and the special fiber is the union of the germ of the coordinate axes with the origin the node. The projection differential is $d\pi = zdw + wdz$. The vertical line bundle $\mathcal{L}$ over V has non vanishing section $z\partial/\partial z - w\partial/\partial w$ and the relative dualizing sheaf $\omega_{V/D}$ has non vanishing section $dz/z - dw/w$.

We consider curve families with noded stable fibers. A proper surjective map $\Pi : \mathcal{C} \to \mathcal{B}$ of analytic spaces is a family of nodal curves provided at each point $\mathbf{c} \in \mathcal{C}$, either Π is smooth with one dimensional fibers, or for an analytic space $\mathcal{S}$ the family is locally analytically equivalent to a locus $zw = f(s)$ in $\mathbb{C}^2 \times \mathcal{S}$ over $\mathcal{S}$ with analytic f vanishing at $\Pi(\mathbf{c})$. The family is a flat family of stable curves [15]. The locus $\{(0, 0)\} \times \mathcal{S}$ is the loci of nodes. We consider families with first order vanishing of f; for a neighborhood of a node the family is analytically equivalent to a Cartesian product of $\pi : V \to D$ and a complex manifold. Stability of fibers is the condition that each component of the nodal complement in a fiber has negative Euler characteristic. Negative Euler characteristic ensures that the automorphism group of a component is finite. The general fiber of a family $\Pi : \mathcal{C} \to \mathcal{B}$ of nodal curves is a smooth Riemann surface (see Fig. 1). The locus of noded curves within the family is a divisor with normal crossings. For a flat family $\Pi : \mathcal{C} \to \mathcal{B}$, the relative dualizing sheaf is isomorphic to the product of canonical bundles $K_{\mathcal{C}} \otimes \Pi^* K_{\mathcal{B}}^{\vee}$.

The relative dualizing sheaf $\omega_{\mathcal{C}/\mathcal{B}}$ provides a generalization of the family of canonical bundles for a family of Riemann surfaces. For k positive, sections of $\omega_{\mathcal{C}/\mathcal{B}}^k$ over open sets of $\mathcal{B}$, generalize families of holomorphic k-differentials for Riemann surfaces. For an analytic space $\mathcal{S}$, a section η of $\omega_{\mathcal{C}/\mathcal{B}}^k$ on a neighborhood of a node $\pi : V \times S \to D \times S$ is given as

$$\eta = \mathbf{f}(z, w, s)\left(\frac{dz}{z} - \frac{dw}{w}\right)^k$$

with $\mathbf{f}$ holomorphic in (z, w, s). For an annulus fiber and the mapping $z = \zeta$, $w = t/\zeta$, $t \neq 0$, into a fiber, the section is given as

$$\eta = \mathbf{f}(\zeta, t/\zeta, s)\left(2\frac{d\zeta}{\zeta}\right)^k. \tag{1}$$