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Lattices and cohomological Mackey functors for finite cyclic p-groups

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Abstract

For a finite cyclic p-group G and a discrete valuation domain R of characteristic 0 with maximal ideal pR the $R[G]$-permutation modules are characterized in terms of the vanishing of first degree cohomology on all subgroups (cf. Theorem A). As a consequence any $R[G]$-lattice can be presented by $R[G]$-permutation modules (cf. Theorem C). The proof of these results is based on a detailed analysis of the category of cohomological G-Mackey functors with values in the category of R-modules. It is shown that this category has global dimension 3 (cf. Theorem E). A crucial step in the proof of Theorem E is the fact that a gentle R-order category (with parameter p) has global dimension less than or equal to 2 (cf. Theorem D).

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1. Introduction

For a Dedekind domain R and a finite group G one calls a finitely generated left $R[G]$-module M an $R[G]$-*lattice*, if M – considered as an R-module – is projective. In this paper we

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focus on the study of $R[G]$-lattice, where R is a discrete valuation domain of characteristic 0 with maximal ideal pR for some prime number p, and G is a finite cyclic p-group. The study of such lattices has a long history and was motivated by a promising result of F.-E. Diederichsen (cf. [5, Thm. 34:31], [6]) who showed that for the finite cyclic group of order p there are precisely three directly indecomposable such lattices up to isomorphism: the trivial $R[G]$-lattice R, the free $R[G]$-lattice $R[G]$, and the augmentation ideal $\omega_{R[G]} = \ker(R[G] \to R)$. A similar finiteness result holds for cyclic groups of order p^2 (cf. [12]). However, for cyclic p-groups of order larger than p^2 there will be infinitely many isomorphism types of such lattices; even worse, in general this classification problem is "wild" (cf. [7,8,11]). If the $R[G]$-lattice M is isomorphic to $R[\Omega]$ for some finite left G-set Ω, M will be called an $R[G]$-*permutation lattice*. The main purpose of this paper is to establish the following characterization of $R[G]$-permutation lattices for finite cyclic p-groups (cf. Corollary 6.7, Proposition 6.8).

Theorem A. *Let R be a discrete valuation domain of characteristic 0 with maximal ideal pR for some prime number p, let G be a finite cyclic p-group, and let M be an $R[G]$-lattice. Then the following are equivalent.*

(i) *M is an $R[G]$-permutation lattice,*
(ii) *$H^1(U, \mathrm{res}^G_U(M)) = 0$ for all subgroups U of G,*
(iii) *M_U is R-torsion free for all subgroups U of G,*

where $M_U = M/\omega_{R[U]}M$ denotes the U-coinvariants of M.

By a result of I. Reiner (cf. [5, Theorem 34.31], [15]), one knows that there are $\mathbb{Z}[C_p]$-lattices satisfying (ii), where C_p is the cyclic group of order p, which are not $\mathbb{Z}[C_p]$-permutation lattices. Hence the conclusion of Theorem A does not hold for the ring $R = \mathbb{Z}$.

Theorem A has a number of interesting consequences which we would like to explain in more detail. For a finite p-group G it is in general quite difficult to decide whether a given $R[G]$-lattice M is indeed an $R[G]$-permutation lattice. A sufficient criterion to the just mentioned problem was given by A. Weiss in [27] for an arbitrary finite p-group G and the ring of p-adic integers $R = \mathbb{Z}_p$. He showed that if for a normal subgroup N of G the $\mathbb{Z}_p[G/N]$-module M^N of N-invariants is a $\mathbb{Z}_p[G/N]$-permutation module, and $\mathrm{res}^G_N(M)$ is a free $\mathbb{Z}_p[N]$-module, then M is a $\mathbb{Z}_p[G]$-permutation module (cf. [13, Chap. 8, Theorem 2.6]). Theorem A extends A. Weiss' result for cyclic p-groups in the following way (cf. Proposition 6.12).

Corollary B. *Let R be a discrete valuation domain of characteristic 0 with maximal ideal pR for some prime number p, let G be a finite cyclic p-group, and let N be a normal subgroup of G. Suppose that the $R[G]$-lattice M is satisfying the following two hypothesis.*

(i) *$\mathrm{res}^G_N(M)$ is an $R[N]$-permutation module, and*
(ii) *M^N is an $R[G/N]$-permutation module.*

Then M is an $R[G]$-permutation module.

Although it seems impossible to describe all isomorphism types of directly indecomposable $R[G]$-lattices, where R is a discrete valuation domain of characteristic 0 with maximal ideal pR and G is a finite cyclic p-group, one can (re)present such lattices in a very natural way (cf. Theorem 6.11).