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Advances in Mathematics 244 (2013) 708–749

ADVANCES IN Mathematics

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# The first variation of the total mass of log-concave functions and related inequalities

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Received 11 July 2012; accepted 10 May 2013
Available online 24 June 2013

Communicated by Erwin Lutwak

**Abstract**

On the class of log-concave functions on $\mathbb{R}^n$, endowed with a suitable algebraic structure, we study the first variation of the total mass functional, which corresponds to the volume of convex bodies when restricted to the subclass of characteristic functions. We prove some integral representation formulae for such a first variation, which suggest to define in a natural way the notion of area measure for a log-concave function. In the same framework, we obtain a functional counterpart of Minkowski's first inequality for convex bodies; as corollaries, we derive a functional form of the isoperimetric inequality, and a family of logarithmic-type Sobolev inequalities with respect to log-concave probability measures. Finally, we propose a suitable functional version of the classical Minkowski's problem for convex bodies, and prove some partial results towards its solution.


*MSC:* primary 26B25; 26D10; 52A20

*Keywords:* Log-concave functions; Convex bodies; Area measure; Isoperimetric inequality; Log-Sobolev inequality; Minkowski's problem

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0001-8708/$ - see front matter 
http://dx.doi.org/10.1016/j.aim.2013.05.015

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## 1. Introduction

This article regards *log-concave* functions defined in $\mathbb{R}^n$, *i.e.* functions of the form

$$f : \mathbb{R}^n \to \mathbb{R}, \qquad f = e^{-u},$$

where $u : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is convex.

In the last decades the interest in log-concave functions has been considerably increasing, strongly motivated by the analogy between these objects and convex bodies (convex compact subsets of $\mathbb{R}^n$).

The first breakthrough in the discovery of parallel behaviours of convex bodies and log-concave functions, was the *Prékopa–Leindler inequality*, named after the two Hungarian mathematicians who proved it in the seventies [20,25–27]. It states that, for any given functions $f, g, h \in L^1(\mathbb{R}^n; \mathbb{R}_+)$ which satisfy, for some $t \in (0, 1)$, the pointwise inequality

$$h\big((1-t)x + ty\big) \geq f(x)^{1-t} g(y)^t \quad \forall x, y \in \mathbb{R}^n,$$

it holds

$$\int_{\mathbb{R}^n} h \geq \left(\int_{\mathbb{R}^n} f\right)^{1-t} \left(\int_{\mathbb{R}^n} g\right)^t. \tag{1.1}$$

Moreover, it was proved by Dubuc in [9] that the equality sign holds in (1.1) if and only if the functions $f$ and $g$ are log-concave and translates, meaning that $f(x) = g(x - x_0)$ for some $x_0 \in \mathbb{R}^n$.

If $K$ and $L$ are measurable subsets of $\mathbb{R}^n$ such that also their Minkowski's combination $(1-t)K + tL$ is measurable, by applying the Prékopa–Leindler inequality with $f, g$ and $h$ equal respectively to the characteristic functions of $K$, $L$ and $(1-t)K + tL$, one obtains

$$V((1-t)K + tL) \geq V(K)^{1-t}\, V(L)^t.$$

This is an equivalent formulation of the classical *Brunn–Minkowski inequality*

$$V((1-t)K + tL)^{1/n} \geq (1-t)V(K)^{1/n} + tV(L)^{1/n}, \tag{1.2}$$

which holds with equality sign if and only if $K$ and $L$ belong to the class $\mathcal{K}^n$ of convex bodies in $\mathbb{R}^n$ and are homothetic, namely they agree up to a translation and a dilation.