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Ideal approximation theory

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Abstract

Let $(\mathcal{A};\mathcal{E})$ be an exact category and $\mathcal{F}\subseteq \mathrm{Ext}$ a subfunctor. A morphism φ in $\mathcal{A}$ is an $\mathcal{F}$-phantom if the pullback of an $\mathcal{E}$-conflation along φ is a conflation in $\mathcal{F}$. If the exact category $(\mathcal{A};\mathcal{E})$ has enough injective objects and projective morphisms, it is proved that an ideal $\mathcal{I}$ of $\mathcal{A}$ is special precovering if and only if there is a subfunctor $\mathcal{F}\subseteq \mathrm{Ext}$ with enough injective morphisms such that $\mathcal{I}$ is the ideal of $\mathcal{F}$-phantom morphisms. A crucial step in the proof is a generalization of Salce's Lemma for ideal cotorsion pairs: if $\mathcal{I}$ is a special precovering ideal, then the ideal cotorsion pair $(\mathcal{I},\mathcal{I}^{\perp})$ generated by $\mathcal{I}$ in $(\mathcal{A};\mathcal{E})$ is complete. This theorem is used to verify: (1) that the ideal cotorsion pair cogenerated by the pure-injective modules of R-Mod is complete; (2) that the ideal cotorsion pair cogenerated by the contractible complexes in the category of complexes $\mathrm{Ch}(R\text{-Mod})$ is complete; and, using Auslander and Reiten's theory of almost split sequences, (3) that the ideal cotorsion pair cogenerated by the Jacobson radical $\mathrm{Jac}(\Lambda\text{-mod})$ of the category Λ-mod of finitely generated representations of an Artin algebra is complete.

MSC: 18E10; 18G15; 18G25; 16G70

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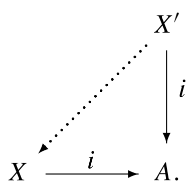
1. Introduction

Let $(\mathcal{A}; \mathcal{E})$ be an exact category [9,14,22]. Given a subcategory $\mathcal{C} \subseteq \mathcal{A}$, define ${}^{\perp}\mathcal{C} \subseteq \mathcal{A}$ to be the subcategory of objects F such that $\mathrm{Ext}(F, C) = 0$ for every $C \in \mathcal{C}$, and define $\mathcal{C}^{\perp}$ dually. A *cotorsion pair* in $(\mathcal{A}; \mathcal{E})$ is a pair $(\mathcal{F}, \mathcal{C})$ of subcategories of $\mathcal{A}$ satisfying $\mathcal{F} = {}^{\perp}\mathcal{C}$ and $\mathcal{C} = \mathcal{F}^{\perp}$. The notion of a cotorsion pair was introduced by Salce ([27], but see [17, Lemma 2.2.6]) in the setting of Ab, the abelian category of abelian groups. In the general setting of an exact category, this notion provides the proper context for the study of precovers and preenvelopes (approximation theory). Expositions of approximation theory for categories of modules may be found in the monographs of Beligiannis and Reiten [4] and Göbel and Trlifaj [17], but cotorsion pairs have also been used to study approximation theory in sheaf categories [13], general Grothendieck categories [15,20], and more abstract exact categories [28].

While the present state of approximation theory tends to stress the importance of objects and subcategories, the purpose of this article is to give morphisms and ideals of categories equal status. If $\mathcal{J}$ is an ideal of $\mathcal{A}$, define ${}^{\perp}\mathcal{J}$ to be the ideal of morphisms i such that $\mathrm{Ext}(i, j) = 0$ for every $j \in \mathcal{J}$, and define $\mathcal{J}^{\perp}$ dually. An *ideal cotorsion pair* in $(\mathcal{A}; \mathcal{E})$ is a pair $(\mathcal{I}, \mathcal{J})$ of ideals of $\mathcal{A}$ satisfying $\mathcal{I} = {}^{\perp}\mathcal{J}$ and $\mathcal{J} = \mathcal{I}^{\perp}$. Our aim is to develop approximation theory for ideal cotorsion pairs in analogy with the approximation theory of cotorsion pairs in an exact category. In contrast to the references above, no completeness assumptions are made on an exact category.

Examples of preenvelopes are the injective and pure-injective envelopes of a module. The existence of flat precovers was conjectured by Enochs [11] and proved in [8]. These are all examples (cf. [12]) of approximations relative to a *subcategory* of the category R-Mod of left R-modules over an associative ring R. But there also exist approximations relative to an ideal. For example, if Λ is an Artin algebra, and Λ-mod denotes the category of finitely presented left Λ-modules, the work of Auslander and Reiten [1] shows that every object $M \in \Lambda$-mod has a cover (resp., envelope) with respect to the ideal $\mathrm{Jac}(\Lambda\text{-mod})$ given by the *Jacobson radical* of Λ-mod. Another example, one that provides the prototype for the present theory, is given by the ideal of phantom morphisms in a module category R-Mod; the existence of phantom covers was proved by the third author [19].

Let $\mathcal{I} \subseteq \mathcal{A}$ be an ideal and A an object of $\mathcal{A}$. An *$\mathcal{I}$-precover* of A is a morphism $i \in \mathcal{I}$, $i : X \to A$, such that any other morphism $i' : X' \to A$ in $\mathcal{I}$ factors through i,



If the category is equipped with an exact structure $(\mathcal{A}; \mathcal{E})$, then an $\mathcal{I}$-precover $i : X \to A$ of A in $\mathcal{A}$ is *special* if it is obtained as the pushout of a conflation η along a morphism $j : Y \to B$ in $\mathcal{I}^{\perp}$: