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Cooperation principle, stability and bifurcation in random complex dynamics

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Abstract

We investigate the random dynamics of rational maps and the dynamics of semigroups of rational maps on the Riemann sphere $\hat{\mathbb{C}}$. We show that regarding random complex dynamics of polynomials, generically, the chaos of the averaged system disappears at any point in $\hat{\mathbb{C}}$, due to the automatic cooperation of the generators. We investigate the iteration and spectral properties of transition operators acting on the space of (Hölder) continuous functions on $\hat{\mathbb{C}}$. We also investigate the stability and bifurcation of random complex dynamics. We show that the set of stable systems is open and dense in the space of random dynamical systems of polynomials. Moreover, we prove that for a stable system, there exist only finitely many minimal sets, each minimal set is attracting, and the orbit of a Hölder continuous function on $\hat{\mathbb{C}}$ under the transition operator tends exponentially fast to the finite-dimensional space U of finite linear combinations of unitary eigenvectors of the transition operator. Combining this with the perturbation theory for linear operators, we obtain that for a stable system constructed by a finite family of rational maps, the projection to the space U depends real-analytically on the probability parameters. By taking a partial derivative of the function of probability of tending to a minimal set with respect to a probability parameter, we introduce a complex analogue of the Takagi function, which is a new concept.

MSC: 37F10; 37H10

Keywords: Random dynamical systems; Random complex dynamics; Rational semigroups; Fractal geometry; Cooperation principle; Noise-induced order; Randomness-induced phenomena

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1. Introduction

In this paper, we investigate the independent and identically-distributed (i.i.d.) random dynamics of rational maps on the Riemann sphere $\hat{\mathbb{C}}$ and the dynamics of rational semigroups (i.e., semigroups of non-constant rational maps where the semigroup operation is functional composition) on $\hat{\mathbb{C}}$.

One motivation for research in complex dynamical systems is to describe some mathematical models on ethology. For example, the behavior of the population of a certain species can be described by the dynamical system associated with iteration of a polynomial $f(z) = az(1-z)$ (cf. [9]). However, when there is a change in the natural environment, some species have several strategies to survive in nature. From this point of view, it is very natural and important not only to consider the dynamics of iteration, where the same survival strategy (i.e., function) is repeatedly applied, but also to consider random dynamics, where a new strategy might be applied at each time step. Another motivation for research in complex dynamics is Newton's method to find a root of a complex polynomial, which often is expressed as the dynamics of a rational map g on $\hat{\mathbb{C}}$ with $\deg(g) \geq 2$, where $\deg(g)$ denotes the degree of g. We sometimes use computers to analyze such dynamics, and since we have some errors at each step of the calculation in the computers, it is quite natural to investigate the random dynamics of rational maps. In various fields, we have many mathematical models which are described by the dynamical systems associated with polynomial or rational maps. For each model, it is natural and important to consider a randomized model, since we always have some kind of noise or random terms in nature. The first study of random complex dynamics was given by J. E. Fornaess and N. Sibony [10]. They mainly investigated random dynamics generated by small perturbations of a single rational map. For research on random complex dynamics of quadratic polynomials, see [4–8,11]. For research on random dynamics of polynomials (of general degrees), see the author's works [30,28,29,31,32].

In order to investigate random complex dynamics, it is very natural to study the dynamics of associated rational semigroups. In fact, it is a very powerful tool to investigate random complex dynamics, since random complex dynamics and the dynamics of rational semigroups are related to each other very deeply. The first study of dynamics of rational semigroups was conducted by A. Hinkkanen and G. J. Martin [14], who were interested in the role of the dynamics of polynomial semigroups (i.e., semigroups of non-constant polynomial maps) while studying various one-complex-dimensional moduli spaces for discrete groups, and by F. Ren's group [12], who studied such semigroups from the perspective of random dynamical systems. Since the Julia set $J(G)$ of a finitely generated rational semigroup $G = \langle h_1, \dots, h_m \rangle$ has "backward self-similarity", i.e., $J(G) = \bigcup_{j=1}^{m} h_j^{-1}(J(G))$ (see [25, Lemma 0.2]), the study of the dynamics of rational semigroups can be regarded as the study of "backward iterated function systems", and also as a generalization of the study of self-similar sets in fractal geometry. For recent work on the dynamics of rational semigroups, see the author's papers [24–32] and [22,23,33,34].

In this paper, by combining several results from [31] and many new ideas, we investigate the random complex dynamics and the dynamics of rational semigroups. In the usual iteration dynamics of a single rational map g with $\deg(g) \geq 2$, we always have a non-empty chaotic part, i.e., in the Julia set $J(g)$ of g, which is a perfect set, we have sensitive initial values and dense orbits. Moreover, for any ball B with $B \cap J(g) \neq \emptyset$, $g^n(B)$ expands as $n \to \infty$. Regarding random complex dynamics, it is natural to ask the following question. Do we have a kind of "chaos" in the averaged system? Or do we have no chaos? How do many kinds of maps in the system interact? What can we say about stability and bifurcation? Since the chaotic phenomena hold even for a single rational map, one may expect that in random dynamics of rational maps,