



Representation theory and homological stability

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Abstract

We introduce the idea of *representation stability* (and several variations) for a sequence of representations V_n of groups G_n . A central application of the new viewpoint we introduce here is the importation of representation theory into the study of homological stability. This makes it possible to extend classical theorems of homological stability to a much broader variety of examples. Representation stability also provides a framework in which to find and to predict patterns, from classical representation theory (Littlewood–Richardson and Murnaghan rules, stability of Schur functors), to cohomology of groups (pure braid, Torelli and congruence groups), to Lie algebras and their homology, to the (equivariant) cohomology of flag and Schubert varieties, to combinatorics (the $(n + 1)^{n-1}$ conjecture). The majority of this paper is devoted to exposing this phenomenon through examples. In doing this we obtain applications, theorems and conjectures.

Beyond the discovery of new phenomena, the viewpoint of representation stability can be useful in solving problems outside the theory. In addition to the applications given in this paper, it is applied by Church–Ellenberg–Farb (in preparation) [20] to counting problems in number theory and finite group theory. Representation stability is also used by Church (2012) [19] to give broad generalizations and new proofs of classical homological stability theorems for configuration spaces on oriented manifolds.

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Contents

1.	Introduction.....	251
2.	Representation stability.....	259
2.1.	Symmetric and hyperoctahedral groups	259
2.2.	The algebraic groups SL_n , GL_n and Sp_{2n}	260
2.3.	Definition of representation stability.....	263
2.4.	Strong and mixed tensor stability	265
3.	Stability in classical representation theory	269
3.1.	Combining and modifying stable sequences	269
3.2.	Reversing the Clebsch–Gordan problem	275
4.	Cohomology of pure braid and related groups.....	277
4.1.	Stability of the cohomology of pure braid groups.....	278
4.2.	Generalized braid groups	282
4.3.	Groups of string motions.....	284
5.	Lie algebras and their homology	285
5.1.	Graded Lie algebras and Lie algebra homology	285
5.2.	Simple representation stability.....	289
5.3.	Applications and examples.....	290
5.4.	The Malcev Lie algebra of the pure braid group.....	295
6.	Homology of the Torelli subgroups of $\text{Mod}(S)$ and $\text{Aut}(F_n)$	296
6.1.	Homology of the Torelli group.....	296
6.2.	Homology of IA_n	298
6.3.	Vanishing and finiteness conjectures for the (co)homology of $\mathcal{I}_{g,1}$ and IA_n	299
7.	Flag varieties, Schubert varieties, and rank-selected posets.....	300
7.1.	Cohomology of flag varieties.....	300
7.2.	Cohomology of Schubert varieties.....	303
7.3.	Rank-selected posets	304
7.4.	The $(n + 1)^{n-1}$ conjecture.....	305
8.	Congruence subgroups, modular representations and stable periodicity.....	306
8.1.	A motivating example.....	307
8.2.	Modular representations of finite groups of Lie type.....	307
8.3.	Stable periodicity and congruence subgroups	308
8.4.	The abelianization of the Torelli group	310
8.5.	Level p mapping class groups	310
	Acknowledgments.....	311
	References	312

1. Introduction

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