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Advances in Mathematics 245 (2013) 250–314

ADVANCES IN Mathematics

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Representation theory and homological stability

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Received 19 January 2012; accepted 17 June 2013
Available online 13 July 2013

Communicated by the Managing Editors of AIM

Abstract

We introduce the idea of *representation stability* (and several variations) for a sequence of representations V_n of groups G_n. A central application of the new viewpoint we introduce here is the importation of representation theory into the study of homological stability. This makes it possible to extend classical theorems of homological stability to a much broader variety of examples. Representation stability also provides a framework in which to find and to predict patterns, from classical representation theory (Littlewood–Richardson and Murnaghan rules, stability of Schur functors), to cohomology of groups (pure braid, Torelli and congruence groups), to Lie algebras and their homology, to the (equivariant) cohomology of flag and Schubert varieties, to combinatorics (the $(n+1)^{n-1}$ conjecture). The majority of this paper is devoted to exposing this phenomenon through examples. In doing this we obtain applications, theorems and conjectures.

Beyond the discovery of new phenomena, the viewpoint of representation stability can be useful in solving problems outside the theory. In addition to the applications given in this paper, it is applied by Church–Ellenberg–Farb (in preparation) [20] to counting problems in number theory and finite group theory. Representation stability is also used by Church (2012) [19] to give broad generalizations and new proofs of classical homological stability theorems for configuration spaces on oriented manifolds.

Keywords: Representation theory; Homological stability; Braid groups

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0001-8708/$ - see front matter
http://dx.doi.org/10.1016/j.aim.2013.06.016

Contents

1. Introduction

In this paper we introduce the idea of *representation stability* (and several variations) for a sequence of representations V_n of groups G_n. A central application of the new viewpoint we introduce here is the importation of representation theory into the study of homological stability. This make it possible to extend classical theorems of homological stability to a much broader variety of examples. Representation stability also provides a framework in which to find and to predict patterns, from classical representation theory (Littlewood–Richardson and Murnaghan rules, stability of Schur functors), to cohomology of groups (pure braid, Torelli and congruence groups), to Lie algebras and their homology, to the (equivariant) cohomology of flag and Schubert varieties, to combinatorics (the $(n+1)^{n-1}$ conjecture). The majority of this paper is devoted to