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Modular forms and special cubic fourfolds

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Abstract

We study the degrees of special cubic divisors on moduli space of cubic fourfolds with at worst ADE singularities. In this paper, we show that the generating series of the degrees of such divisors is a level three modular form.

Keywords: Special cubic fourfolds; Modular form; Heegner divisors; Eisenstein series

1. Introduction

The classical Noether–Lefschetz locus for degree d hypersurfaces in $\mathbb{P}^3$ is the locus in the Hilbert scheme space $\mathbb{P}^{\binom{d+3}{3}-1}$ where the Picard rank is greater than one. For $d \geq 4$, the Noether–Lefschetz loci are known to be a countable union of proper subvarieties of $\mathbb{P}^{\binom{d+3}{3}-1}$ by Griffiths and Harris [8]. A natural question is to find the degrees of these subvarieties. For quartic surfaces in $\mathbb{P}^3$, Maulik and Pandharipande [19] showed that the Noether–Lefschetz loci are divisors and the degrees of these divisors are the Fourier coefficients of certain modular forms.

In higher dimensional cases, cubic fourfolds have received a lot of attention since their period map behaves quite nicely. Specifically, the period domain for cubic fourfolds is a bounded symmetric domain of type IV and the global Torelli theorem holds (cf. [25,26]).

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The analogues of Noether–Lefschetz loci for surfaces are the loci of special cubic fourfolds studied by Hassett [9]. A smooth cubic fourfold X in $\mathbb{P}^5$ is *special* of discriminant $d > 6$ if it contains an algebraic surface S, and the discriminant of the saturated lattice spanned by h^2 and $[S]$ in $H^4(X, \mathbb{Z})$ is d, where $h = c_1(\mathcal{O}_X(1))$. The Zariski closure of the collection of such cubic fourfolds forms an irreducible divisor $\mathcal{C}_d$ in the moduli space $\mathcal{M}$ (cf. [15]) of cubic fourfolds with at worst isolated ADE singularities and it is nonempty if and only if $d \equiv 0, 2 \bmod 6$. We interpret $\mathcal{C}_6$ as the set of singular cubics in $\mathcal{M}$. The special cubic divisor C_d in the Hilbert scheme $\mathbb{P}^{55}$ of cubic hypersurfaces is the lift of $\mathcal{C}_d$ (see Section 2 for more details). In the present paper, we study the degree of special cubic divisors C_d. Our main result is the following.

Theorem 1. *Let $\Theta(q) = -2 + \sum_{d>2}^{\infty} \deg(C_d) q^{\frac{d}{6}}$ be the generating series for the degrees of the special cubic divisors. Then $\Theta(q)$ is a modular form of weight 11 and level 3 with expansion:*

$$\begin{aligned}\Theta(q) &= -\alpha^{11}(q) + 162\alpha^8(q)\beta(q) + 91854\alpha^5(q)\beta^2(q) + 2204496\alpha^2(q)\beta^3(q)\\ &\quad - \alpha^{11}\left(q^{\frac{1}{3}}\right) + 66\alpha^8\left(q^{\frac{1}{3}}\right)\beta\left(q^{\frac{1}{3}}\right) - 1386\alpha^5\left(q^{\frac{1}{3}}\right)\beta^2\left(q^{\frac{1}{3}}\right)\\ &\quad + 9072\alpha^2\left(q^{\frac{1}{3}}\right)\beta^3\left(q^{\frac{1}{3}}\right)\\ &= -2 + 192q + 3402q^{\frac{4}{3}} + 196272q^2 + 915678q^{\frac{7}{3}} + \cdots\end{aligned}$$

where

$$\alpha(q) = 1 + 6\sum_{n\geq 1} q^n \sum_{d|n}\left(\frac{d}{3}\right) \quad \text{and} \quad \beta(q) = \sum_{n\geq 1} q^n \sum_{d|n}(n/d)^2\left(\frac{d}{3}\right) \tag{1.1}$$

are level three modular forms of weight 1 and 3 that generate the space of modular forms with respect to the group $\Gamma_0(3)$ (see Section 3.2). Here, $\left(\frac{d}{3}\right)$ denotes the Legendre symbol.

The approach to Theorem 1 is via the result of Borcherds [2] and Kudla–Millson [14]. The degrees of C_d are the Fourier coefficients of a vector-valued modular form. As in [19], the Noether–Lefschetz numbers are related to the reduced Gromov–Witten (GW) invariants of K3 surfaces. We hope there is a similar GW-theory interpretation of $\deg(C_d)$.

Outline of the paper. In Section 2, we review some classical result on cubic fourfolds and describe the special cubic divisors from an arithmetic perspective. Section 3 is the central section of this paper. We recap Borcherds' work on Heegner divisors to prove the modularity of a vector-valued generating series of $\deg(C_d)$. This vector-valued modular form can be expressed explicitly in terms of some well-known modular forms. The proof of our main theorem is presented in the last section.

After posting our paper on the arXiv server, we learned from Atanas Iliev that, in forthcoming work, Atanas Iliev, Emanuel Scheidegger, and Ludmil Katzarkov in [11] have independently proved Theorem 3 using a different basis of the space of vector-valued modular forms.

2. Special cubic fourfolds and Heegner divisors

In this section, we review some results on special cubic divisors and the relation with Heegner divisors associated to a signature $(m, 2)$ lattice, which is defined from an arithmetic perspective. Throughout the paper, we denote by $L^\vee$ the dual of a lattice L and $O(L)$ its associated orthogonal group.