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Multiply-refined enumeration of alternating sign matrices

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Abstract

Four natural boundary statistics and two natural bulk statistics are considered for alternating sign matrices (ASMs). Specifically, these statistics are the positions of the 1's in the first and last rows and columns of an ASM, and the numbers of generalized inversions and -1's in an ASM. Previously-known and related results for the exact enumeration of ASMs with prescribed values of some of these statistics are discussed in detail. A quadratic relation which recursively determines the generating function associated with all six statistics is then obtained. This relation also leads to various new identities satisfied by generating functions associated with fewer than six of the statistics. The derivation of the relation involves combining the Desnanot–Jacobi determinant identity with the Izergin–Korepin formula for the partition function of the six-vertex model with domain-wall boundary conditions.

Keywords: Alternating sign matrices; Six-vertex model with domain-wall boundary conditions; Desnanot–Jacobi identity

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1. Introduction

A major focus of attention throughout the history of alternating sign matrices (ASMs) has simply been the derivation of results related to their exact enumeration. Such results typically state that the number of ASMs which satisfy specific conditions, such as having prescribed values of certain statistics or being invariant under certain symmetry operations, is given by an explicit formula or generating function, is equal to the number of combinatorial objects of some other variety satisfying specific conditions, or is equal to a number which arises from a particular physical model.

A few examples of such results, with references to conjectures and initial proofs, are as follows: a formula for the total number of ASMs of any fixed size but with no further conditions applied (Mills, Robbins and Rumsey [108,109], Zeilberger [152], and Kuperberg [97]); a formula for the number of ASMs with a prescribed boundary row or column (Mills, Robbins and Rumsey [108,109], and Zeilberger [153]); formulae for numbers of ASMs invariant under certain natural symmetry operations (Robbins [129,130], Kuperberg [98], Okada [117], and Razumov and Stroganov [125,126]); equalities between numbers of certain ASMs and numbers of certain totally symmetric self-complementary plane partitions (Mills, Robbins and Rumsey [110], and