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On the classification of easy quantum groups

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Abstract

In 2009, Banica and Speicher began to study the compact quantum groups G with $S_n \subseteq G \subseteq O_n^+$ whose intertwiner spaces are induced by some partitions. These so-called *easy* quantum groups have a deep connection to combinatorics. We continue their work on classifying these objects, by introducing some new examples of easy quantum groups. In particular, we show that the six easy groups O_n, S_n, H_n, B_n, $S_n{}'$ and $B_n{}'$ split into seven cases O_n^+, S_n^+, H_n^+, B_n^+, $S_n'^{\,+}$, $B_n'^{\,+}$ and $B_n^{\#+}$ on the side of free easy quantum groups. Also, we give a complete classification in the half-liberated and in the nonhyperoctahedral case.

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0. Introduction

In 2009, Banica and Speicher [7] initiated the classification program for easy quantum groups. These are compact quantum groups G such that $S_n \subseteq G \subseteq O_n^+$, where S_n is the symmetric group and O_n^+ is the (free) orthogonal quantum group, constructed by Wang [27]. If, in addition, the intertwiner space of G (which determines the quantum group G by Woronowicz, [33]) is induced by certain partitions, the quantum group G is called *easy*. In other words, in order to understand the quantum groups in between the symmetric quantum group S_n^+ (by Wang [28]) and O_n^+, or – more generally – in between S_n and O_n^+, Banica and Speicher restrict their attention to those that are given by some underlying combinatorial data.

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Furthermore, the class of easy quantum groups appears very natural, since the intertwiner spaces of the (classical) groups S_n and O_n are given by all partitions resp. by all pair partitions, whereas their free analogues S_n^+ and O_n^+ correspond to all noncrossing partitions resp. to all noncrossing pair partitions. Thus, the connection of the quantum groups in between S_n and O_n^+ to combinatorics is not at all artificial, quite the contrary, it is an intrinsic background that needs to be exploited. Besides, it is a well known feature in free probability theory that the step from "commutative" to "free" is reflected by passing from all partitions to the noncrossing ones. So, also from this perspective it is worth to study this effect in the present case.

The easy quantum groups have been investigated by Banica, Speicher, Curran, Bichon, Collins and some others in a couple of papers (see for instance [7,4,5,2]). Further operator algebraic aspects of some of the easy quantum groups have been studied in [25,8,9,17]. The link between easy quantum groups and free probability theory has been discovered by Köstler and Speicher, [21]. Their noncommutative version of the De Finetti theorem hints at the deep connection between co-actions of (easy) quantum groups and free probability theory, which lead to further work by Banica, Curran and Speicher (see also [13–16,6]).

In the case of classical easy quantum groups (which are in fact groups) $S_n \subseteq G \subseteq O_n$, the classification is complete and has been done in [7]. There are exactly six easy groups: O_n, H_n, S_n, B_n, S_n' and B_n'. In the case of free easy quantum groups $S_n^+ \subseteq G \subseteq O_n^+$, one quantum group was missing on the list in [7]. We very much follow the proof of Banica and Speicher and we show that there are exactly seven free easy quantum groups (also called *free orthogonal quantum groups*): O_n^+, H_n^+, S_n^+, B_n^+, $S_n'^{+}$, $B_n'^{+}$ and $B_n^{\#+}$. The subtlety in the free case is that the group B_n' splits into two cases $B_n'^{+}$ and $B_n^{\#+}$, whereas all other easy groups and free easy quantum groups are in a natural one-to-one correspondence. We describe this phenomenon to a certain extent although it is still not completely understood. Nevertheless, this shows again that the "free" world is somewhat richer than the commutative one.

In [7,4], further examples of easy quantum groups are given, the half-liberated versions O_n^* and H_n^* of O_n^+ resp. of H_n^+ as well as the (infinite) hyperoctahedral series $H_n^{(s)}$ and $H_n^{[s]}$. We extend the list by a half-liberated version $B_n^{\#*}$ of $B_n^{\#+}$ and a further one. We also prove that there are exactly three half-liberated easy quantum groups besides the hyperoctahedral series $H_n^{(s)}$. Hence, the classification is complete in the classical, in the free and in the half-liberated case. However, the complete classification of all easy quantum groups still remains open.

The paper is organized as follows. In Section 1 we give a short introduction into the concept of the classification of easy quantum groups. For details on this, we refer to the initial paper by Banica and Speicher [7]. In Section 2 we focus on the classification of the free easy quantum groups $S_n^+ \subseteq G \subseteq O_n^+$ following the path of [7]. We formulate it in a language of generating partitions, which might help in the future work on the classification. The main results of this section are the description of all possible categories of noncrossing partitions and the proof that there are exactly seven of them, which has partly been done by [7].

A conceptual explanation for the difference of the two quantum versions of B_n' is given in Section 5, where we study the C^*-algebraic level of these objects. We show that the C^*-algebras associated to $B_n'^{+}$ and $B_n^{\#+}$ may be constructed out of the one of B_n^+ by a tensor product resp. a free product with $C^*(\mathbb{Z}_2)$. Likewise, $S_n'^{+}$ is given by a tensor product of S_n^+ with $C^*(\mathbb{Z}_2)$. We use a result on the K-theory of O_n^+ by Voigt [26] to compute the K-theory of B_n^+, $B_n'^{+}$ and $B_n^{\#+}$. The computation of the K-groups for S_n^+, $S_n'^{+}$ and H_n^+ has to be left open. Finally, we prove that none of the full C^*-algebras corresponding to the seven free easy quantum groups is exact, if $n \geq 5$.