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The discriminant of a system of equations

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Abstract

We introduce and study the discriminant of a system of polynomial equations with indeterminate coefficients, generalizing a number of known special cases, such as the sparse resultant and the A-determinant, and inheriting their nice properties. One such property is the fact that the discriminant is always a hypersurface in the space of systems of equations (which seems unexpected when compared to the existence of dual defect toric varieties). This fact reflects the combinatorics of a certain partial order relation on the set of tuples of faces of a tuple of polytopes (the Newton polytopes of the equations in our case), generalizing the poset of faces of one polytope. This combinatorics is the main challenge in our work, and can be reduced to a certain fact of tropical geometry, which resembles the geometry of the discriminants that we study. As a sample application of our version of the multivariate discriminant, we prove that atypical fibers of a generic polynomial map are distinguished by their Euler characteristics, and the bifurcation set of such a map is a hypersurface, whose degree we explicitly compute.

Keywords: Discriminant; Purity; Dual defect; Tropical geometry; Polytope; Bifurcation set

1. Introduction

Which polynomial in the coefficients of a system of algebraic equations should be called its discriminant? We prove a package of facts that provide a possible answer. Let us call a system typical, if the homeomorphic type of its set of solutions does not change as we perturb its (non-zero) coefficients. The set of all atypical systems turns out to be a hypersurface in the space of all

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systems of k equations in $n \geq k - 1$ variables, whose monomials are contained in k given finite sets (this holds unless some p of the equations essentially depend on less than $p-1$ variables, in which case the set of atypical systems is the closure of the set of consistent systems, i.e. simply the resultant variety). The hypersurface B of atypical systems is the union of two well-known strata: the set of all systems that have a singular solution (this stratum is conventionally called the discriminant) and the set of all systems, whose principal part is degenerate (they can be regarded as systems with a singular solution at infinity). None of these two strata is a hypersurface in general, and codimensions of their components have not been fully understood yet (e.g. dual defect toric varieties are not classified), so the purity of dimension of their union seems somewhat surprising.

A generic system of equations in an irreducible component B_i of the hypersurface B always differs from a typical system by the Euler characteristic of its set of solutions. Regarding the difference of these two Euler characteristics as the multiplicity of B_i, we turn B into an effective divisor, whose equation we call the Euler discriminant of a system of equations by the following reasons. Firstly, it vanishes exactly at those systems that have a singular solution (possibly at infinity). Secondly, despite its topological definition, it admits a simple linear-algebraic formula for its computation, and a positive formula for its Newton polytope. Thirdly, it interpolates many classical objects and inherits many of their nice properties: for $k = n + 1$, it is the sparse resultant (defined by vanishing on consistent systems of equations); for $k = 1$, it is the principal A-determinant (defined as the sparse resultant of the polynomial and its partial derivatives); as we specialize the indeterminate coefficients of our system to be polynomials of a new parameter, the Euler discriminant turns out to be preserved under this base change, similarly to discriminants of deformations. This allows, for example, to specialize our results to generic polynomial maps: the bifurcation set of a dominant polynomial map, whose components are generic linear combinations of given monomials, is always a hypersurface, and a generic atypical fiber of such a map differs from a typical one by its Euler characteristic.

The study of the Euler discriminant of a system of equations can be reduced to the combinatorics of the Newton polytopes of the equations. In particular, we have to study a certain natural partial order relation on the set of tuples of compatible faces of a tuple of polytopes, reflecting adjacencies of the faces in a more subtle way than the poset of faces for the Minkowski sum of the polytopes does. We are far from finishing the study of this combinatorics, however we manage to extract what we need for Euler discriminants, reducing the problem to the following fact: the stable intersection of a tropical fan with the boundary of a polytope in the ambient space is a hypersurface in the fan. This combinatorial geometry is the main challenge in our work.

1.1. Degenerate systems

For a finite set $H \subset \mathbb{Z}^n$, we study the space $\mathbb{C}[H]$ of all *Laurent polynomials* $h(x) = \sum_{a\in H} c_a x^a$, where x^a stands for the monomial $x_1^{a_1} \dots x_n^{a_n}$, the coefficient c_a is a complex number, and the polynomial h is considered as a function $(\mathbb{C} \setminus 0)^n \to \mathbb{C}$. For a linear function $v : \mathbb{Z}^n \to \mathbb{Z}$, denote the intersection of H with the boundary of the affine half-space $H + \{v < 0\}$ by H^v, and the highest v-degree component $\sum_{a\in H^v} c_a x^a$ by h^v (if $v = 0$, then we set $H^0 = H$ and $h^0 = h$).

In what follows, we denote a collection of finite sets $A_0, \dots, A_k$ in $\mathbb{Z}^n$ by A, the space $\mathbb{C}[A_0] \oplus \dots \oplus \mathbb{C}[A_k]$ by $\mathbb{C}[A]$, consider its element $f = (f_0, \dots, f_k) \in \mathbb{C}[A]$ as a map $(\mathbb{C} \setminus 0)^n \to \mathbb{C}^{k+1}$, and denote $(f_0^v, \dots, f_k^v)$ by f^v.