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Perfect forms, K-theory and the cohomology of modular groups

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Abstract

For $N = 5$, 6 and 7, using the classification of perfect quadratic forms, we compute the homology of the Voronoï cell complexes attached to the modular groups $SL_N(\mathbb{Z})$ and $GL_N(\mathbb{Z})$. From this we deduce the rational cohomology of those groups and some information about $K_m(\mathbb{Z})$, when $m = 5$, 6 and 7.

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Contents

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1. Introduction

Let $N \geqslant 1$ be an integer and let $SL_N(\mathbb{Z})$ be the modular group of integral matrices with determinant one. Our goal is to compute its cohomology groups with trivial coefficients, i.e. $H^q\big(SL_N(\mathbb{Z}), \mathbb{Z}\big)$. The case $N = 2$ is well-known and follows from the fact that $SL_2(\mathbb{Z})$ is the amalgamated product of two finite cyclic groups ([29,7], II.7, Ex.3, p. 52). The case $N = 3$ was done in [31]: for any $q > 0$ the group $H^q\big(SL_3(\mathbb{Z}), \mathbb{Z}\big)$ is killed by 12. The case $N = 4$ has been studied by Lee and Szczarba in [19]: modulo 2, 3 and 5-torsion, the cohomology group $H^q\big(SL_4(\mathbb{Z}), \mathbb{Z}\big)$ is trivial whenever $q > 0$, except that $H^3\big(SL_4(\mathbb{Z}), \mathbb{Z}\big) = \mathbb{Z}$. In Theorem 7.3 below, we solve the cases $N = 5$, 6 and 7.

For these calculations we follow the method of [19], i.e. we use the perfect forms of Voronoï. Recall from [34,20] that a perfect form in N variables is a positive definite real quadratic form h on $\mathbb{R}^N$ which is uniquely determined (up to a scalar) by its set of integral minimal vectors. Voronoï proved in [34] that there are finitely many perfect forms of rank N, modulo the action of $SL_N(\mathbb{Z})$. These are known for $N \leqslant 8$ (see Section 2 below).

Voronoï used perfect forms to define a cell decomposition of the space X_N^* of positive real quadratic forms, the kernel of which is defined over $\mathbb{Q}$. This cell decomposition (cf. Section 2) is invariant under $SL_N(\mathbb{Z})$, hence it can be used to compute the equivariant homology of X_N^* modulo its boundary. On the other hand, this equivariant homology turns out to be isomorphic