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Decompositions of polyhedral products for shifted complexes

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Abstract

The conjecture of Bahri, Bendersky, Cohen and Gitler (2010) [3] on wedge decompositions of polyhedral products of shifted complexes is settled affirmatively. As a corollary, it is proved that the homotopy type of the complement of a coordinate subspace arrangement associated with a shifted complex, tensored with $\mathbb{R}^r$ for any $r \geq 1$, has the homotopy type of a wedge of spheres.

Keywords: Polyhedral product; Homotopy decomposition; Shifted complex; Subspace arrangement

1. Introduction

Throughout the paper, spaces and maps mean compactly generated weak Hausdorff spaces having non-degenerate base points and base point preserving maps.

Let us begin with defining polyhedral products. Let K be an abstract simplicial complex on the index set $[m] = \{1, \dots, m\}$, where we assume that the empty set is a simplex of K for our convention. Let $(\underline{X}, \underline{A})$ be a collection of pairs of spaces indexed by $[m]$, say

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$(\underline{X}, \underline{A}) = \{(X_i, A_i)\}_{i\in[m]}$. For a simplex σ of K, we put

$$(\underline{X}, \underline{A})^\sigma = Y_1 \times \cdots \times Y_m, \quad \text{where } Y_i = \begin{cases} X_i & i \in \sigma \\ A_i & i \notin \sigma. \end{cases}$$

The polyhedral product (or the generalized moment–angle complex) of $(\underline{X}, \underline{A})$ with respect to K is defined as

$$\mathcal{Z}_K(\underline{X}, \underline{A}) = \bigcup_{\sigma\in K} (\underline{X}, \underline{A})^\sigma,$$

where the union is taken in $X_1 \times \cdots \times X_m$.

Polyhedral products (with respect to the boundary of a simplex) first appeared in the work of Porter [12] in which higher order Whitehead products are defined as the natural maps between certain polyhedral products. After this work, polyhedral products have been studied in homotopy theory along several directions. Recently, in the work of Davis and Januszkiewicz [5], the special polyhedral product $\mathcal{Z}_K(\underline{D}^2, \underline{S}^1)$, called the moment–angle complex of K, was found to play a fundamental role in their theory of, so-called, quasi-toric manifolds (cf. [4]), which is a topological analogue of theory of toric varieties, where $(\underline{D}^2, \underline{S}^1)$ is the m-copies of (D^2, S^1). Since then, many mathematicians have been studying polyhedral products in a variety of directions, not only in homotopy theory. See [1], [3,6–9], [11], for example. In this paper, we are particularly interested in wedge decompositions of polyhedral products. Let us recall two results on wedge decompositions of polyhedral products; one is due to Grbić and Theriault [9] and the other is due to Bahri, Bendersky, Cohen and Gitler [3].

To state the result of Grbić and Theriault [9], we introduce special simplicial complexes called shifted complexes.

Definition 1.1. An abstract simplicial complex K is called shifted if its vertex set is given a total order satisfying for any simplex $\sigma \in K$ and a vertex $v \in \sigma$, $(\sigma - v) \cup w$ is also a simplex of K whenever a vertex w satisfies $v < w$.

Remark 1.2. In the above definition of shifted complexes, the order of vertices is opposite to the one in [3,9], which is convenient for us and is just a notational difference.

The most elementary examples of shifted complexes are skeleta of simplices. Other examples will be given in Section 5. We now state the result of Grbić and Theriault [9].

Theorem 1.3 (*Grbić and Theriault [9]*)**.** *If K is a shifted complex, $\mathcal{Z}_K(\underline{D}^2, \underline{S}^1)$ has the homotopy type of a wedge of spheres.*

Remark 1.4. The proof of Grbić and Theriault [9] heavily relies on the fact that S^1 has the classifying space, and then it cannot be applied to a general collection $(C\underline{X}, \underline{X}) = \{(CX_i, X_i)\}_{i\in[m]}$.

A few years after the work of Grbić and Theriault [9], Bahri, Bendersky, Cohen and Gitler [3] gave another wedge decomposition of a suspension of a polyhedral product, which is a simple generalization of the standard homotopy equivalence $\Sigma(X \times Y) \simeq \Sigma X \vee \Sigma Y \vee \Sigma(X \wedge Y)$. Although they considered a general polyhedral product $\mathcal{Z}_K(\underline{X}, \underline{A})$, what we are interested in is the special polyhedral product $\mathcal{Z}_K(C\underline{X}, \underline{X})$, and then we here state the result for $\mathcal{Z}_K(C\underline{X}, \underline{X})$ only. Let us set notation. For a non-empty subset I of the vertex set of a simplicial complex K, let K_I denote the induced subcomplex on I (or the full subcomplex on I), that is, K_I is the