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On entire solutions of an elliptic system modeling phase separations

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Abstract

We study the qualitative properties of a limiting elliptic system arising in phase separation for Bose–Einstein condensates with multiple states:

$$\begin{cases} \Delta u = uv^2 & \text{in } \mathbb{R}^n, \\ \Delta v = vu^2 & \text{in } \mathbb{R}^n, \\ u, v > 0 & \text{in } \mathbb{R}^n. \end{cases}$$

When $n = 1$, we prove uniqueness of the one-dimensional profile. In dimension 2, we prove that stable solutions with linear growth must be one-dimensional. Then we construct entire solutions in $\mathbb{R}^2$ with polynomial growth $| x |^d$ for any positive integer $d \geq 1$. For $d \geq 2$, these solutions are not one-dimensional. The construction is also extended to multi-component elliptic systems.

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1. Introduction and main results

Consider the following two-component Gross–Pitaevskĭi system

$$-\Delta u + \alpha u^3 + \Lambda v^2 u = \lambda_1 u \quad \text{in } \Omega, \tag{1.1}$$

$$-\Delta v + \beta v^3 + \Lambda u^2 v = \lambda_2 v \quad \text{in } \Omega, \tag{1.2}$$

$$u > 0, \qquad v > 0 \quad \text{in } \Omega, \tag{1.3}$$

$$u = 0, \qquad v = 0 \quad \text{on } \partial\Omega, \tag{1.4}$$

$$\int_\Omega u^2 = N_1, \qquad \int_\Omega v^2 = N_2, \tag{1.5}$$

where $\alpha, \beta, \Lambda > 0$ and Ω is a bounded smooth domain in $\mathbb{R}^n$. Such systems are associated with mixtures of Bose–Einstein condensates cfr, [3,24,25]. Solutions of (1.1)–(1.5) can be regarded as critical points of the energy functional

$$E_\Lambda(u, v) = \int_\Omega \left(|\nabla u|^2 + |\nabla v|^2\right) + \frac{\alpha}{2}u^4 + \frac{\beta}{2}v^4 + \frac{\Lambda}{2}u^2 v^2, \tag{1.6}$$

on the space $(u, v) \in H_0^1(\Omega) \times H_0^1(\Omega)$ with constraints

$$\int_\Omega u^2 dx = N_1, \qquad \int_\Omega v^2 dx = N_2. \tag{1.7}$$

The eigenvalues λ_j's are Lagrange multipliers with respect to (1.7). Both eigenvalues $\lambda_j = \lambda_{j,\Lambda}$, $j = 1, 2$, and eigenfunctions $u = u_\Lambda$, $v = v_\Lambda$ depend on the parameter Λ. As the parameter Λ tends to infinity, the two components tend to separate their supports. In order to investigate the basic rules of phase separations in this system one needs to understand the asymptotic behavior of (u_Λ, v_Λ) as $\Lambda \to +\infty$.

We shall assume that the solutions (u_Λ, v_Λ) of (1.1)–(1.5) are such that the associated eigenvalues $\lambda_{j,\Lambda}$'s are uniformly bounded, together with their energies $E_\Lambda(u_\Lambda, v_\Lambda)$. Then, as $\Lambda \to +\infty$, there is weak convergence (up to a subsequence) to a limiting profile (u_∞, v_∞) which formally satisfies

$$\begin{cases} -\Delta u_\infty + \alpha u_\infty^3 = \lambda_{1,\infty} u_\infty & \text{in } \Omega_u, \\ -\Delta v_\infty + \beta v_\infty^3 = \lambda_{2,\infty} v_\infty & \text{in } \Omega_v, \end{cases} \tag{1.8}$$

where $\Omega_u = \{x \in \Omega : u_\infty(x) > 0\}$ and $\Omega_v = \{x \in \Omega : v_\infty(x) > 0\}$ are positivity domains composed of finitely disjoint components with positive Lebesgue measure, and each $\lambda_{j,\infty}$ is the limit of $\lambda_{j,\Lambda}$'s as $\Lambda \to \infty$ (up to a subsequence).

There is a large literature about this type of questions. Effective numerical simulations for (1.8) can be found in [5,6,14]. Chang–Lin–Lin–Lin [14] proved pointwise convergence of (u_Λ, v_Λ) away from the interface $\Gamma \equiv \{x \in \Omega : u_\infty(x) = v_\infty(x) = 0\}$. In Wei–Weth [32] the uniform equicontinuity of (u_Λ, v_Λ) is established, while Noris–Tavares–Terracini–Verzini [26] proved the uniform-in-Λ Hölder continuity of (u_Λ, v_Λ); afterwards, Wang [31] proved localized uniform Hölder estimates. The regularity of the nodal set of the limiting profile has been investigated in [12,30] and in [17]: it turns out that the limiting pair $(u_\infty(x), v_\infty(x))$ is the positive and the negative pair (w^+, w^-) of a solution of the equation $-\Delta w + \alpha(w^+)^3 - \beta(w^-)^3 = \lambda_{1,\infty} w^+ - \lambda_{2,\infty} w^-$.