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Bucolic complexes

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Abstract

We introduce and investigate bucolic complexes, a common generalization of systolic complexes and CAT(0) cubical complexes. They are defined as simply connected prism complexes satisfying some local combinatorial conditions. We study various approaches to bucolic complexes: from graph-theoretic and topological perspectives, as well as from the point of view of geometric group theory. In particular, we characterize bucolic complexes by some properties of their 2-skeleta and 1-skeleta (that we call bucolic graphs), by which several known results are generalized. We also show that locally-finite bucolic complexes are contractible, and satisfy some nonpositive-curvature-like properties.

Keywords: CAT(0) cubical and systolic complexes; Median and bridged graphs; Gated amalgamation; Cartesian product; Prism complexes; Retracts; Fixed points; Asphericity

1. Introduction

CAT(0) cubical complexes and systolic (simplicial) complexes constitute two classes of polyhedral complexes that have been intensively explored over last decades. Both CAT(0) cubical

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and systolic complexes exhibit various properties typical for spaces with different types of nonpositive curvature. Hence groups of isomorphisms of such complexes provide numerous examples of groups with interesting properties. Both CAT(0) cubical complexes and systolic complexes can be nicely characterized via their 1- and 2-skeleta. It turns out that their 1-skeleta – median graphs and bridged graphs – are intensively studied in various areas of discrete mathematics (see Section 3 for related results and references).

In this article we introduce a notion of *bucolic complexes*—polyhedral complexes being a common generalization of CAT(0) cubical [24,37], systolic [17,25,29], and weakly systolic [32] complexes, and we initiate a regular study of them. Analogously to CAT(0) cubical and systolic complexes, bucolic complexes are defined as simply connected prism complexes satisfying some local combinatorial conditions (see Section 2.4 for details). Our main result on bucolic complexes is the following characterization via their 1- and 2-skeleta (see Section 2 for explanations of all the notions involved).

Theorem 1. *For a prism complex* $\mathbf{X}$*, the following conditions are equivalent:*

(i) $\mathbf{X}$ *is a bucolic complex;*
(ii) *the* 2*-skeleton* $\mathbf{X}^{(2)}$ *of* $\mathbf{X}$ *is a connected and simply connected triangle–square flag complex satisfying the wheel, the* 3*-cube, and the* 3*-prism conditions;*
(iii) *the* 1*-skeleton* $G(\mathbf{X})$ *of* $\mathbf{X}$ *is a connected weakly modular graph that does not contain induced subgraphs of the form* $K_{2,3}, W_4$*, and* W_4^-*, i.e.,* $G(\mathbf{X})$ *is a bucolic graph not containing infinite hypercubes.*

Moreover, if $\mathbf{X}$ *is a connected flag prism complex satisfying the wheel, the cube, and the prism conditions, then the universal cover* $\widetilde{\mathbf{X}}$ *of* $\mathbf{X}$ *is bucolic.*

As an immediate corollary we obtain an analogous characterization (Corollary 3 in Section 5) of *strongly bucolic complexes* – the subclass of bucolic complexes containing products of systolic complexes but not all weakly systolic complexes (see Section 2.4 for details). The condition (iii) in the above characterization is a global condition—weak modularity concerns balls of arbitrary radius; cf. Section 2. Thus the theorem – and in particular the last assertion – may be seen as a local-to-global result concerning polyhedral complexes. It is an analogue of the Cartan–Hadamard theorem appearing in various contexts of non-positive-curvature: CAT(0) spaces [7], Gromov hyperbolic spaces [24], systolic and weakly systolic complexes [29,32].

The 1-skeleta of CAT(0) cubical complexes are exactly the median graphs [17,36] which constitute a central graph class in metric graph theory (see [6] and the references therein). In the literature there are numerous structural and other characterizations of median graphs. In particular, median graphs are the retracts of hypercubes [2], and can be obtained via so-called iterated gated amalgamations from cubes [28,39]. The general framework of fiber-complemented graphs was introduced in [14,15] and allows to prove such decomposition and retraction results. From this perspective, *bucolic graphs* are exactly the fiber-complemented graphs whose elementary gated subgraphs are weakly-bridged; more precisely, the 1-skeleta of bucolic complexes admit the following characterization.

Theorem 2. *For a graph* $G = (V, E)$ *not containing infinite cliques, the following conditions are equivalent:*

(i) G *is a retract of the (weak) Cartesian product of weakly bridged (respectively, bridged) graphs;*
(ii) G *is a weakly modular graph not containing induced* $K_{2,3}$*,* W_4*, and* W_4^- *(respectively,* $K_{2,3}$*,* W_4^-*,* W_4*, and* W_5*), i.e.,* G *is a bucolic (respectively, strongly bucolic) graph;*