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Orlov spectra as a filtered cohomology theory ☆

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Abstract

This paper presents a new approach to the dimension theory of triangulated categories by considering invariants that arise in the pretriangulated setting.

Keywords: Differential graded categories; Homological dimension; Triangulated categories; A-infinity modules

1. Introduction

In [17], Rouquier gave several results on the dimension theory of triangulated categories. Following this paper, Orlov computed the dimension of the derived category of coherent sheaves on an arbitrary smooth curve and found it to equal one in [16]. Orlov then advanced a more general perspective on dimension theory by defining the spectrum of a triangulated category, now called the Orlov spectrum, which includes the generation times of all strong generators. The relevance of strong generators in triangulated categories and their connection to algebraic geometry was thoroughly established in the seminal paper [3] by Bondal and Van den Bergh. As the Orlov spectrum compares the generation times amongst all strong generators, it serves as a more nuanced invariant than dimension.

In the important recent work [1] of Ballard, Favero and Katzarkov, gaps in the Orlov spectrum were found to depend on the existence of algebraic cycles. To further this line of reasoning,

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they stated the following conjectures which link large gaps in the Orlov spectrum to birational invariants.

Conjecture 1.1. *Let X be a smooth algebraic variety. If* $\langle \mathcal{A}_1, \ldots, \mathcal{A}_n \rangle$ *is a semi-orthogonal decomposition of* $\mathcal{T}$*, then the length of any gap in* $D^b(X)$ *is at most the maximal Rouquier dimension amongst the* $\mathcal{A}_i$.

Conjecture 1.2. *Let X be a smooth algebraic variety. If* $\mathcal{A}$ *is an admissible subcategory of* $D^b(X)$*, then the length of any gap of* $\mathcal{A}$ *is at most the maximal length of any gap of* $D^b(X)$*. Conversely, if* $\mathcal{A}$ *has a gap of length at least s, then so does* $D^b(X)$.

These have many important corollaries connecting birational geometry to triangulated categories and their Orlov spectrum. We recall again from [1] two such results.

Corollary 1.3. *Suppose Conjectures 1.1 and 1.2 hold. Let X and Y be birational smooth proper varieties of dimension n. The category,* $D^b(X)$*, has a gap of length n or* $n-1$ *if and only if* $D^b(Y)$ *has a gap of the same length i.e. the gaps of length greater than* $n-2$ *are a birational invariant.*

Corollary 1.4. *Suppose Conjectures 1.1 and 1.2 hold. If X is a rational variety of dimension n, then any gap in* $D^b(X)$ *has length at most* $n-2$.

Establishing a procedure for computing the Orlov spectrum of $D^b(X)$ would also allow us to pursue, for example, the following.

Conjecture 1.5. *Let X be a generic smooth four dimensional cubic. Then the gap of the spectra of the derived category of this cubic is equal to two.*

From the considerations above, this conjecture implies that generic smooth four dimensional cubic is not rational, a standing question in algebraic geometry.

While the triangulated setting serves as an accessible model for homological invariants, it is generally accepted that triangulated categories are inadequate for giving a natural characterization of homotopy theory for derived categories. Instead of working in this setting, it is advisable to lift to a pretriangulated category, or $(\infty, 1)$-category framework, where several constructions are more natural [15,7]. In this paper, we study the Orlov spectra of triangulated categories by lifting to pretriangulated DG or A_∞-categories.

When the category $\mathcal{T}$ is strongly generated by a compact object G, we upgrade several classical results in dimension theory of abelian categories to the pretriangulated setting and find that the natural filtration given by the bar construction plays a determining role in the calculus of dimension. Indeed, if G is such a generator, using a result of Lefèvre-Hasegawa, we can regard $\mathcal{T}$ as the homotopy category of perfect modules over an A_∞ algebra $A_G = \mathrm{Hom}^*(G, G)$. In addition to being a DG category, the category of perfect A_∞ modules over A_G is enhanced over filtered chain complexes, where the filtration is obtained through the bar construction. This filtration descends to the triangulated level. The first main result, Theorem 3.12, in this paper is that the generation time of a strong generator G equals the maximal length of this filtration.

Theorem 1.6. *The generation time of* $G \in \mathcal{T}$ *equals the supremum over all* $M, N \in A_G\text{-}mod_\infty$ *of the lengths of* $\mathrm{Hom}_{A_G\text{-}mod_\infty}(M, N)$ *with respect to the filtration induced by the bar construction.*

As a result, we develop a filtered cohomology theory which yields the generation times that occur in Orlov spectra. The lengths referred to in this theorem are those of the filtrations induced