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On deformations of triangulated models

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Abstract

This paper is the first part of a project aimed at understanding deformations of triangulated categories, and more precisely their dg and A_∞ models, and applying the resulting theory to the models occurring in the Homological Mirror Symmetry setup. In this first paper, we focus on models of derived and related categories, based upon the classical construction of twisted objects over a dg or A_∞-algebra. For a Hochschild 2 cocycle on such a model, we describe a corresponding "curvature compensating" deformation which can be entirely understood within the framework of twisted objects. We unravel the construction in the specific cases of derived A_∞ and abelian categories, homotopy categories, and categories of graded free qdg-modules. We identify a purity condition on our models which ensures that the structure of the model is preserved under deformation. This condition is typically fulfilled for homotopy categories, but not for unbounded derived categories.

Keywords: A infinity categories; Triangulated categories; Derived categories; Deformations

1. Introduction

A by now standard philosophy in non-commutative algebraic geometry is that non-commutative spaces can be represented by suitable categorical models based upon sheaf categories and their derived categories in algebraic geometry. Among models we can roughly distinguish between "small" (corresponding morally to "algebraic") and "large" (corresponding morally to "geometric") models. The large models typically occur as module or sheaf type

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categories over the small models. The primordial example of a small model is a ring A, and its associated large model is its module category $\mathsf{Mod}(A)$. In the case of a commutative ring A, there is an intermediate geometric object $\mathrm{Spec}(A)$ for which $\mathsf{Mod}(A) \cong \mathsf{Qch}(\mathrm{Spec}(A))$.

In understanding the relation between commutative objects and their non-commutative counterparts, a crucial role is being played by Hochschild cohomology. From the ring case, Hochschild cohomology is known to describe first order non-commutative deformations, and it turns out that for various more complicated models, natural notions of Hochschild cohomology exist which fulfill the same role. On the side of small models, a notion of Hochschild cohomology for schemes [34] describes deformations into non-commutative schemes based upon twisted presheaves [21].

On the side of large models, a first important class is given by abelian categories (generalizing module and sheaf categories). An intrinsic first order deformation theory for abelian categories was developed in [24], and a notion of Hochschild cohomology was defined in function of controlling this theory [23]. This notion further coincides with some other natural definitions as shown in [11]. The deformation theory of abelian categories has some desirable relations to the classical Gerstenhaber deformation theory of algebras. First of all, for an algebra A, there is an equivalence

$$\mathrm{Def}_{alg}(A) \longrightarrow \mathrm{Def}_{ab}(\mathsf{Mod}(A)) : B \longmapsto \mathsf{Mod}(B) \tag{1}$$

between algebra deformations of A and abelian deformations of $\mathsf{Mod}(A)$. More generally, deformations of Grothendieck categories remain Grothendieck. If a Grothendieck category further has a representation as an additive sheaf category with respect to a topology which can be understood on an underlying set-theoretic level, it can be "tracked" through the deformation process and we obtain structural results for deformations (see [6] for the case of quasi-coherent sheaf categories over suitable projective schemes).

It is known that a lot of geometric information is actually encoded in the *derived* categories of schemes, and it is often possible to model derived categories using combinatorial tools like quivers. More generally, it is always possible to model the derived category of sufficiently nice schemes using dg algebras as "small" models [27,3,12]. These facts motivate the derived approach to non-commutative geometry, with enhanced triangulated categories rather than abelian categories as fundamental models for non-commutative spaces. Here, enhancements are given by dg or A_∞-categories, and thus, they come with a natural notion of Hochschild cohomology.

In line with the higher story, a fundamental question is to understand in which way this Hochschild cohomology can be interpreted as describing certain first order deformations. In the case of derived categories of abelian categories, a first step in this direction was undertaken in [22]. However, in that paper, only linear (fixed object) deformations are considered, leading to an incomplete picture. To understand the problem, we first return to abelian deformations. It is clear that whereas k-algebra deformations themselves generalize straightforwardly to linear deformations of k-linear categories with many objects (simply by keeping the object set fixed and deforming the Hom modules), this is not the correct deformation concept for the abelian module categories for by (1), their object set changes, and so will the object set of their derived category. This is directly related to the fact that when we look at the obstruction theory for deforming an individual object $C \in \mathcal{C}$ to a deformation $\mathcal{D}$ of $\mathcal{C}$, there is an obstruction against lifting in $\mathrm{Ext}^2_{\mathcal{C}}(C, C)$ and if this obstruction vanishes, the freedom for lifting is given by $\mathrm{Ext}^1_{\mathcal{C}}(C, C)$ (well known for modules—see [20] for a treatment in the setup of abelian categories). Hence, obstructions are responsible for the vanishing of some objects under deformation, whereas the