Available online at www.sciencedirect.com

SciVerse ScienceDirect

Advances in Mathematics 231 (2012) 605–623

ADVANCES IN Mathematics

www.elsevier.com/locate/aim

Connected components of definable groups and o-minimality I

Annalisa Conversano[a,*], Anand Pillay[b]

[a] *University of Konstanz, Fach. Mathematik und Statistik, D-78457 Konstanz, DE, Germany*
[b] *University of Leeds, School of Mathematics, LS2 9JT Leeds, UK*

Received 26 July 2011; accepted 31 May 2012
Available online 29 June 2012

Communicated by H. Jerome Keisler

Abstract

We give examples of definable groups G (in a saturated model, sometimes o-minimal) such that $G^{00} \neq G^{000}$, yielding also new examples of "non G-compact" theories. We also prove that for G definable in a (saturated) o-minimal structure, G has a "bounded orbit" (i.e. there is a type of G whose stabilizer has bounded index) if and only if G is definably amenable, giving a positive answer to a conjecture of Newelski and Petrykowski in this *special case* of groups definable in o-minimal structures.

Keywords: Definable group; o-minimality; Amenability; Bounded orbit

1. Introduction and preliminaries

In this paper, groups definable in o-minimal and closely related structures are studied, partly for their own sake and partly as a "testing ground" for general conjectures. Given a $\emptyset$-definable group G in a saturated structure $\overline{M}$, $G^{00}_{\emptyset}$ is the smallest subgroup of G of bounded index which is type-definable over $\emptyset$, and $G^{000}_{\emptyset}$ is the smallest subgroup of G of bounded index which is $\mathrm{Aut}(\overline{M})$-invariant. In o-minimal structures and more generally theories with *NIP*, these "connected components" remain unchanged after naming parameters and so are just referred to

* Correspondence to: Massey University Albany, IIMS, Private Bag 102904, North Shore City 0745, New Zealand.
E-mail addresses: conversa@mail.dm.unipi.it (A. Conversano), pillay@maths.leeds.ac.uk (A. Pillay).

0001-8708/$ - see front matter
doi:10.1016/j.aim.2012.05.022

as G^{00} and G^{000}. In any case $G^{00}_{\emptyset}$ and $G^{000}_{\emptyset}$ are "definable group" analogues of the groups of *KP*-strong automorphisms and Lascar strong automorphisms, respectively, of a saturated structure. The relationship between these definable group and automorphism group notions is explored in [10]. Although examples were given in [2] where the strong automorphism groups differ, until now no example was known where $G^{000}_{\emptyset} \neq G^{00}_{\emptyset}$. In this paper (Section 3) we give a "natural" example: G is simply a saturated elementary extension of $\widetilde{SL_2(\mathbb{R})}$ (the universal cover of $SL_2(\mathbb{R})$) in the language of groups. G is *not* actually definable in an *o*-minimal structure, but we give another closely related example which is. In any case the two-sorted structure consisting of G and a principal homogeneous space for G is now a (natural) example of a "non G-compact" structure (or theory) i.e. where the group of Lascar strong automorphisms is properly contained in the group of *KP*-strong automorphisms.

Another fruitful theme in recent years has been the generalization of stable group theory outside the stable context. The *o*-minimal case has been important and there is now a good understanding of "definably compact" groups from this point of view; for example they are definably amenable, "generically stable for measure", and G is dominated by G/G^{00}. It should be remarked that for *any* group G definable in a (saturated) *o*-minimal structure, G/G^{00}, equipped with the logic topology, is a compact Lie group [1]. In the current paper we try to go beyond the definably compact setting, motivated partly by questions of Newelski and Petrykowski. In [11], definable groups G with "finitely satisfiable generics" (which include definably compact groups in *o*-minimal structures) were shown to be definably amenable by lifting the Haar measure on G/G^{00} to a left invariant Keisler measure on G, making use of a global *generic type* p, whose stabilizer is G^{00}. We guess this encouraged Petrykowski to suggest that if a definable group G (in any structure) has a global type whose stabilizer has "bounded index" then G is definably amenable. Note that a left invariant *type* is a special case of a left invariant Keisler measure, so trivially if there is a global type with stabilizer G then G is definably amenable. In any case, in Section 4 we confirm Petrykowski's conjecture when G is definable in an *o*-minimal structure, as well as raise questions about the nature of types with bounded orbit in the *o*-minimal and more generally *NIP* environment.

In Section 2 of the paper we give a rather basic decomposition theorem (implicit in the literature) for groups in *o*-minimal structures, which is useful for understanding the issues around definable amenability and bounded orbits, as well as G^{00} and G^{000} (although Section 3 can be more or less read independently of Section 2). We introduce and discuss the notion of G having a "good decomposition" (Definition 2.7). The *o*-minimal examples where $G^{00} \neq G^{000}$ will be also examples where good decomposition fails, although good decomposition does hold for algebraic groups.

In a sequel [5] to the current paper we will give a systematic account of G^{00}, G^{000} as well as the quotient G^{00}/G^{000}, for groups G definable in *o*-minimal structures. The decomposition theorem (2.6), refinements of it, as well as the notion of good decomposition, will play major roles.

In general T will denote a complete theory, M an arbitrary model of T, and G a group definable in M. We sometimes work in a sufficiently saturated and homogeneous model $\overline{M}$ of T, in which case "small" or "bounded" essentially means of cardinality strictly less than the degree of saturation of $\overline{M}$, but we will make the meaning more precise later in the paper. *Definability* usually means with parameters, and we say A-definable to mean definable with parameters from A for A a subset of M. When we talk about *o*-minimal theories we will mean *o-minimal expansions of the theory RCF of real closed fields* (and we leave it for later or to others