Available online at www.sciencedirect.com

SciVerse ScienceDirect

ADVANCES IN Mathematics

Advances in Mathematics 231 (2012) 1068–1084

www.elsevier.com/locate/aim

Extracting long basic sequences from systems of dispersed vectors

Jarno Talponen

Aalto University, Institute of Mathematics, P.O. Box 11100, FI-00076 Aalto, Finland

Received 24 January 2011; accepted 26 June 2012
Available online 20 July 2012

Communicated by the Managing Editors of AIM

Abstract

We study Banach spaces satisfying some geometric or structural properties involving tightness of transfinite sequences of nested linear subspaces. These properties are much weaker than WCG and closely related to Corson's property (C). Given a transfinite sequence of normalized vectors, which is dispersed or null in some sense, we extract a subsequence which is a biorthogonal sequence, or even a weakly null monotone basic sequence, depending on the setting. The Separable Complementation Property is established for spaces with an M-basis under rather weak geometric properties. We also consider an analogy of the Baire Category Theorem for the lattice of closed linear subspaces.

MSC: primary 46B20; 46B26; secondary 46Bxx; 46M40

Keywords: Basic sequence; Transfinite sequence; Minimal sequence; Biorthogonal system; WCG; WLD; SCP; CSP; SD; Property C; Dispersed sequence; Strongly dispersed; Baire category; Inverse limit

1. Introduction

Many Banach spaces have a rich structure of projections (see e.g. [17]). This can be applied in classifying spaces, and it is also often easier to work in separable complemented fragments of the space. For example, spaces with a Schauder basis admit a very convenient structure of projections, especially if the basis is unconditional. On the other hand, spaces with very few projections (even few operators, see e.g. [21]), like hereditarily indecomposable spaces, are

E-mail address: talponen@iki.fi.

0001-8708/$ - see front matter
doi:10.1016/j.aim.2012.06.008

currently an object of wide interest. This is partly due to several dichotomies about the existence of projections, roughly stating that if there are reasonably many projections, then there already exists a basic sequence with some required properties, like unconditionality (see e.g. [2,11]).

This paper deals with nonseparable Banach spaces enjoying some structural properties much weaker than reflexivity. These properties involve tightness conditions for transfinite chains of nested linear subspaces. We aim to show that spaces with such chains admit plenty of bounded linear projections. As a result, we will establish the separable complementation property, the existence of basic sequences, or other such properties depending on the setting. As the title of the paper suggests, the main problem here is to extract a transfinite basic sequence or a biorthogonal sequence from a net of vectors, which is in some sense far from being constant.

The following theorem is typical here and it is a kind of prototypical consequence of the main results.

Theorem 1.1. *Let* X *be a Banach space and* $\{x_\alpha\}_{\alpha<\kappa} \subset \mathrm{X}$ *a normalized sequence, where* κ *is an uncountable regular cardinal.*

(1) *If* $\{x_\alpha\}_{\alpha<\kappa}$ *is weakly null, then there is a subsequence* $\{x_{\alpha_\sigma}\}_{\sigma<\kappa}$ *which forms a monotone basic sequence.*
(2) *If* X *has Corson's property* (C) *and* $\{x_\alpha\}_{\alpha<\kappa}$ *is dispersed (resp. strongly dispersed), then there is a subsequence* $\{x_{\alpha_\sigma}\}_{\sigma<\kappa}$ *which forms a bounded biorthogonal sequence (resp. a monotone basic sequence).*

Moreover, each nonseparable Plichko space contains an uncountable monotone basic sequence.

This result will be given in more generality, and we will shortly provide the definition of dispersed and strongly dispersed sequences. Actually, the principle behind the above statement (1) has been essentially known since the work of Bessaga–Pelczynski in 1958 [4], at least in the countable setting, and it is a natural example of the type of phenomena studied here. We will also study, motivated by the constructions of the projections, the intersections of a certain kind of fat subspaces.

Recall that the unit ball of a reflexive Banach space is weakly compact, which implies that each normalized sequence has a cluster point with respect to the weak topology. A similar conclusion holds for sequences of length ω_1 should the unit ball be weakly Lindelöf. Thus, above we increased the length of the sequence while weakening the hypothesis. The possibility for this kind of trade-off is typical for Banach spaces. This is mostly due to the fact that Banach spaces are always countably tight in the norm and weak topologies, and more importantly, their *duals* are often countably tight in the ω^*-topology, or usually at least something similar holds. So, even though the weak clustering of *countable* sequences is not necessarily common in general Banach spaces, the weak clustering of *uncountable* sequences (of vectors, subspaces, etc.) is typical and can be ensured by imposing rather weak geometric or structural conditions. Thus the applications of countable and uncountable combinatorics differ considerably in Banach spaces.

The following hypothesis for Banach spaces X becomes useful here:

- For any uncountable, regular cardinal κ each nested sequence $\{A_\alpha\}_{\alpha<\kappa}$ of closed affine subspaces of X has non-empty intersection.

No infinite-dimensional space satisfies the above condition if we consider countable sequences instead. However, the above hypothesis follows readily if the space in question is Lindelöf in the weak topology, e.g. a WCG space. Some other, related tightness conditions for subspace chains will be considered as well. Interestingly, in the last section it turns out that these conditions can