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Kohnen's limit process for real-analytic Siegel modular forms[☆]

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Abstract

Kohnen introduced a limit process for Siegel modular forms that produces Jacobi forms. He asked if there is a space of real-analytic Siegel modular forms such that skew-holomorphic Jacobi forms arise via this limit process. In this paper, we initiate the study of harmonic skew-Maass–Jacobi forms and harmonic Siegel–Maass forms. We improve a result of Maass on the Fourier coefficients of harmonic Siegel–Maass forms, which allows us to establish a connection to harmonic skew-Maass–Jacobi forms. In particular, we answer Kohnen's question in the affirmative.

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Keywords: Siegel–Maass forms; Differential operators; Fourier series expansions; Fourier–Jacobi expansion

1. Introduction

Jacobi forms occur in the Fourier expansion of Siegel modular forms of degree 2, a fact that played an important part in the proof of the Saito–Kurokawa conjecture (see Maass [19–21],

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Andrianov [1], Zagier [32], and Eichler and Zagier [10]). The theory of Jacobi forms has grown enormously since then leading to beautiful applications in many areas of mathematics and physics. Several of these applications rely on real-analytic Jacobi forms, and it has been necessary to investigate such forms in detail (see Skoruppa [29,30], Berndt and Schmidt [2], and Pitale [27], the first and the third author [5], and more recently [4]). For instance, the real-analytic Jacobi forms in Zwegers [33] are examples of harmonic Maass–Jacobi forms [4] that are absolutely vital to the theory of mock theta functions. These real-analytic Jacobi forms also impact the theory of Donaldson invariants of $\mathbb{CP}^2$ that are related to gauge theory (see for example Göttsche and Zagier [13], Göttsche et al. [12], and Malmendier and Ono [24]), and they emerge in recent work on the Mathieu moonshine (see for example Eguchi et al. [9]).

The interplay of holomorphic Jacobi forms and holomorphic Siegel modular forms is well understood, but the analogous situation for real-analytic forms is still mysterious and only partial progress has been made. For example, current work of Dabholkar et al. [7] on quantum black holes and mock modular forms features mock Jacobi forms (which can also be viewed as holomorphic parts of harmonic Maass–Jacobi forms [4,5]) that occur as Fourier coefficients of meromorphic Siegel modular forms. Kohnen [15,16] suggests a completely different approach to connect real-analytic Jacobi forms and Siegel modular forms. We use Kohnen's approach to shed more light on the relation of Jacobi forms and Siegel modular forms in the real-analytic world. Let F be a real-analytic Siegel modular form of degree 2 with Fourier–Jacobi expansion

$$F(Z) = \sum_{m \in \mathbb{Z}} \phi_m(\tau, z, y')\, e^{2\pi i\, m x'}, \tag{1}$$

where throughout the paper, $Z = \begin{pmatrix} \tau & z \\ z & \tau' \end{pmatrix} \in \mathbb{H}_2$ (the Siegel upper half space of degree 2) with $\tau = x + iy$, $z = u + iv$, and $\tau' = x' + iy'$. In general, ϕ_m is not a Jacobi form due to the dependence on y'. However, in the special case that F in (1) is Maass' [22] nonholomorphic Siegel–Eisenstein series of degree 2 and of type $(\frac{1}{2}, k - \frac{1}{2})$, Kohnen [15,16] employs the limit

$$\mathcal{L}(\phi_m) := \lim_{\delta \to \infty} e^{\frac{\delta}{2}}\, e^{2\pi m \frac{v^2}{y}}\, \phi_m\left(\tau, z, \frac{\delta}{4\pi m} + \frac{v^2}{y}\right) \quad (m > 0) \tag{2}$$

to produce skew-holomorphic Jacobi forms of weight k and index m. Naturally, he asks if there is a space of real-analytic Siegel modular forms such that the limit (2) always yields skew-holomorphic Jacobi forms. Note also that if F is a holomorphic Siegel modular form of weight k, then (2) gives precisely the m-th Fourier–Jacobi coefficient of F, i.e., a holomorphic Jacobi form of weight k and index m.

In this paper, we consider the space $\widehat{\mathbb{M}}_k$ of harmonic Siegel–Maass forms of weight k (see Definition 1), which are real-analytic Siegel modular forms of degree 2 and of type $(\frac{1}{2}, k - \frac{1}{2})$ that are annihilated by the matrix-valued Laplace operator $\Omega_{\frac{1}{2}, k-\frac{1}{2}}$ (defined in (4)). Recall that in the degree one case, Bruinier's and Funke's [6] operator ξ_k maps harmonic weak Maass forms of weight k to weakly-holomorphic modular forms of weight $2 - k$, and the kernel of the map ξ_k consists of weakly-holomorphic modular forms of weight k. In (6) we define the corresponding operator $\xi^{(2)}_{\frac{1}{2}, k-\frac{1}{2}}$ for Siegel–Maass forms, which provides a duality between the weights k and $3 - k$ (analogous to the situation of the Jacobi forms in Section 3 and in [5]), and forms in the kernel are analogs of "holomorphic" Siegel–Maass forms. In Section 3, we introduce the space $\widehat{\mathbb{J}}^{sk}_{k,m}$ of harmonic skew-Maass–Jacobi forms of weight k and index m (see Definition 2), which contains the space $J^{sk}_{k,m}$ of skew-holomorphic Jacobi forms of weight k and index m. We use