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The Andrews–Sellers family of partition congruences

Peter Paule, Cristian-Silviu Radu *

Research Institute for Symbolic Computation (RISC), Johannes Kepler University, A-4040 Linz, Austria

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Abstract

In 1994, James Sellers conjectured an infinite family of Ramanujan type congruences for 2-colored Frobenius partitions introduced by George E. Andrews. These congruences arise modulo powers of 5. In 2002 Dennis Eichhorn and Sellers were able to settle the conjecture for powers up to 4. In this article, we prove Sellers' conjecture for all powers of 5. In addition, we discuss why the Andrews–Sellers family is significantly different from classical congruences modulo powers of primes.

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Keywords: Generalized Frobenius partitions; Sellers' conjecture; Partition congruences of Ramanujan type

1. Introduction

1.1. Sellers' conjecture

In his 1984 Memoir [4], Andrews introduced two families of partition functions, $\phi_k(m)$ and $c\phi_k(m)$, which he called generalized Frobenius partition functions. In this paper, we restrict our attention to generalized 2-colored Frobenius partitions. Their generating function reads as follows [4, (5.17)]:

$$\sum_{m=0}^{\infty} c\phi_2(m)q^m = \prod_{n=1}^{\infty} \frac{1-q^{4n-2}}{(1-q^{2n-1})^4(1-q^{4n})}. \tag{1}$$

* Corresponding author.
E-mail addresses: Peter.Paule@risc.uni-linz.ac.at (P. Paule), Silviu.Radu@risc.uni-linz.ac.at, sradu@risc.jku.at (C.-S. Radu).

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Among numerous properties of generalized Frobenius partitions, Andrews also considered congruences of various kinds. For example, he noted and proved [4, p. 28, Cor. 10.1] that

$$c\phi_2(5n+3) \equiv 0 \pmod 5, \; n \geq 0.$$

In 1994, Sellers [20] conjectured that for all integers $n \geq 0$ and $\alpha \geq 1$ one has

$$c\phi_2(5^\alpha n + \lambda_\alpha) \equiv 0 \pmod{5^\alpha}, \tag{2}$$

where λ_α is defined to be the smallest positive integer such that

$$12\lambda_\alpha \equiv 1 \pmod{5^\alpha}. \tag{3}$$

In his joint paper with Eichhorn and Sellers [10] this conjecture was proved for the cases $\alpha = 1, 2, 3, 4$. In this paper, we settle Sellers' conjecture for all α in the spirit of Watson [21].

In addition, we want to highlight the following aspect: at the first glance the congruences (2) seem to fit the standard pattern of Ramanujan type congruences, and one would expect that standard methods would apply in a straightforward manner. But it turns out that a basic feature of such approaches is missing here; namely, ℓ-adic convergence to zero of sequences formed by the application of U-operators to Atkin basis functions. This, we feel, is the reason why Sellers' conjecture has remained open for more than fifteen years.

More information on this, together with some history, is given in the rest of this section and in Section 6. In short, we were able to recover ℓ-adic zero convergence by the introduction of a new type of subspaces of modular functions which behave well under the action of the U-operators. These subspaces, found by computer experiments, came as a perfect surprise to us. They have a simple but interesting description, (42) and (43), and seem to be completely new.

1.2. Ramanujan's congruences

Infinite families of congruences like (2) were first observed for $p(n)$, the number of partitions of n, by Ramanujan [18] in 1919 where he conjectured that for $\ell \in \{5, 7, 11\}$ and $\alpha \geq 1$:

$$p(\ell^\alpha n + \mu_{\alpha,\ell}) \equiv 0 \pmod{\ell^\alpha}, \; n \geq 0, \tag{4}$$

where $\mu_{\alpha,\ell}$ is defined to be the smallest positive integer such that $24\mu_{\alpha,\ell} \equiv 1 \pmod{\ell^\alpha}$. Watson [21] proved the conjecture for $\ell = 5$ and a suitably modified version for $\ell = 7$; thirty years later Atkin [5] settled the $\ell = 11$ case. Concerning Ramanujan's role consult [7].

To put Sellers' conjecture (2) into context, some further remarks on history and background of such identities seem to be in place. First of all, for $\alpha = 1$ Ahlgren and Boylan [2] proved that (4) holds only for $\ell = 5, 7, 11$. This achievement settles a question of Ramanujan and is one of the few results on the non-existence of partition congruences. For the Andrews–Sellers family an analogous result was proved only recently by Dewar [8]; namely that $c\phi_2(2n+1) \equiv 0 \pmod 2$ and $c\phi_2(5n+3) \equiv 0 \pmod 5$ (proved by Andrews [4]) are the only Ramanujan congruences for two-colored generalized Frobenius partitions. Generally, Ramanujan congruences are congruences of the form $\phi(\ell n + \lambda) \equiv 0 \pmod \ell, n \geq 0$, where ℓ is a prime. Concerning congruences not being of Ramanujan type, landmark results due to Ono [16] and Ahlgren [1] say that there are infinitely many of them of the form $p(an+b) \equiv 0 \pmod{\ell^\alpha}$. For generalized two-colored Frobenius partitions analogous results are not yet known.