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Π_1^1-conservation of combinatorial principles weaker than Ramsey's theorem for pairs

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Abstract

We study combinatorial principles weaker than Ramsey's theorem for pairs over the RCA_0 (recursive comprehension axiom) system with Σ_2^0-bounding. It is shown that the cohesiveness (COH), ascending and descending sequence (ADS), and chain/antichain (CAC) principles are all Π_1^1-conservative over Σ_2^0-bounding. In particular, none of these principles proves Σ_2^0-induction.

Keywords: Reverse mathematics; Π_1^1-conservation; RCA_0; Σ_2^0-bounding; Combinatorial principles; Ascending/descending sequence; Chain/antichain; Cohesiveness

1. Introduction

In this paper, we take up the question of exactly how powerful the infinitary combinatorial principles that come from Ramsey theory are. Our particular interest in this investigation is in characterizing the consequences that these principles have for the finite sets. The use of infinitary methods to come to finitary conclusions is intriguing wherever it appears, and our study here is part of a larger investigation into understanding how the infinite sheds light on the finite.

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We approach this question semantically, so we will be thinking recursion theoretically in the seemingly exotic context of nonstandard models of fragments of arithmetic. These nonstandard structures are at the heart of the matter, since one can only argue semantically about arithmetical consequences by considering all possible models of arithmetic.

Our point of departure is the system RCA_0 which we take as our base theory throughout this paper. RCA_0 consists of the usual first-order axioms for arithmetic operations and Σ^0_1-induction relative to parameters, together with the second-order recursive comprehension scheme

$$\forall x[\varphi(x) \leftrightarrow \neg\psi(x)] \to \exists X\, \forall x[x \in X \leftrightarrow \varphi(x)],$$

where φ and ψ are Σ^0_1-formulas also with parameters (we will refer to formulas like this as Δ^0_1-formulas). Now, fix $\mathcal{M} = \langle M, \mathbb{X}, +, \times, 0, 1\rangle$ to be a model of RCA_0, where $\mathbb{X}$ is the collection of subsets of M in $\mathcal{M}$. Ramsey's theorem for pairs (RT^2_2) states that any coloring F in $\mathbb{X}$ of the two-element subsets $\{x, y\}$ of M using only two colors has an $\mathcal{M}$-infinite monochromatic subset, i.e. an $\mathcal{M}$-infinite $A \in \mathbb{X}$ all of whose two-element subsets are assigned the same color by F. This set A is said to be homogeneous for the coloring F, or simply F-homogeneous. It is known that RT^2_2 is not provable in RCA_0. The strength of RT^2_2 in the context of subsystems of second-order arithmetic has been a subject of major interest in reverse mathematics over the past few decades.

Closely related to RT^2_2, and intuitively a more controlled coloring scheme, is the stable Ramsey theorem for pairs (SRT^2_2): if for every $x \in M$, all but finitely many $\{x, y\}$'s have the same color, then there is an $\mathcal{M}$-infinite homogeneous set in $\mathcal{M}$. SRT^2_2 is also known to be unprovable from RCA_0. The proof theoretic strength of these two combinatorial principles has been investigated by various authors. Cholak et al. [1] showed that SRT^2_2, and hence RT^2_2 as first established by Hirst [8], implies the Σ^0_2-bounding principle $B\Sigma^0_2$ (an induction scheme whose strength is known to lie strictly between Σ^0_1- and Σ^0_2-induction [10]), and that RT^2_2 is Π^1_1-conservative over RCA_0 together with the Σ^0_2-induction scheme $I\Sigma^0_2$, i.e. any Π^1_1-statement that is provable in $\mathsf{RT}^2_2 + \mathsf{RCA}_0 + I\Sigma^0_2$ is already provable in the system $\mathsf{RCA}_0 + I\Sigma^0_2$. It follows immediately that any subsystem of $\mathsf{RT}^2_2 + \mathsf{RCA}_0 + I\Sigma^0_2$ (such as replacing RT^2_2 by SRT^2_2) is Π^1_1-conservative over $\mathsf{RCA}_0 + I\Sigma^0_2$.

There are several outstanding open problems relating to RT^2_2 and SRT^2_2, which provided the motivation for the problems studied in this paper. We list three of these: (1) whether over RCA_0, RT^2_2 is strictly stronger than SRT^2_2; (2) whether RT^2_2 or even SRT^2_2 proves $I\Sigma^0_2$, given that they already imply $B\Sigma^0_2$, and (3) whether RT^2_2, or even SRT^2_2, is Π^1_1-conservative over $\mathsf{RCA}_0 + B\Sigma^0_2$.

While these questions remain unsolved, similar or related questions for principles weaker than RT^2_2 or SRT^2_2 have been studied with some degree of success. First of all, Cholak et al. [1] introduced the principle COH and showed the equivalence of RT^2_2 with $\mathsf{COH} + \mathsf{SRT}^2_2$ over the system RCA_0.[1] COH states that every array coded in $\mathcal{M}$ has a set in the model cohesive for the array (see Section 3 for the definition). Since COH is provable from RT^2_2, $\mathsf{COH} + I\Sigma^0_2$ is Π^1_1-conservative over $I\Sigma^0_2$. Secondly, Hirschfeldt and Shore [6] investigated two principles which they demonstrated to be strictly weaker than RT^2_2: the chain and antichain principle (CAC), which states that every infinite partially ordered set coded in $\mathcal{M}$ has an infinite chain or antichain

[1] The original proof of this equivalence in [1] is incorrect; see [2].