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Homogenization for a class of integral functionals in spaces of probability measures

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Abstract

We study the homogenization of a class of actions with an underlying Lagrangian $\mathcal{L}$ defined on the set of absolutely continuous paths in the Wasserstein space $\mathcal{P}_p(\mathbb{R}^d)$. We introduce an appropriate topology on this set and obtain the existence of a Γ-limit of the rescaled Lagrangians. Our main goal is to provide a representation formula for these Γ-limits in terms of the effective Lagrangians. This allows us to study not only the "convexity properties" of the effective Lagrangian, but also the differentiability properties of its Legendre transform restricted to constant functions. For the case $d > 1$ we obtain partial results in terms of an effective Lagrangian defined on $L^p((0,1)^d; \mathbb{R}^d)$. Our study provides a way of computing the limit of a family of metrics on the Wasserstein space. The results of this paper can also be applied to study the homogenization of variational solutions of the one-dimensional Vlasov–Poisson system, as well as the asymptotic behavior of calibrated curves (Fathi (2003) [6], Gangbo and Tudorascu (2010) [12]). Whereas our study for the one-dimensional case covers a large class of Lagrangians, that for the higher dimensional case is concerned with special Lagrangians such as the ones obtained by regularizing the potential energy of the d-dimensional Vlasov–Poisson system.

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1. Introduction

Let us consider a continuous path defined on the set of probability measures on the phase space $\mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d) : t \in [0,T] \to f_t \in \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d)$ and consider rescaled actions defined on such paths

$$\mathbf{A}^\epsilon(f) := \frac{1}{2}\int_0^T \int_{\mathbb{R}^d\times\mathbb{R}^d} |v|^2 f_t(\mathrm{d}x,\mathrm{d}v) - \int_{\mathbb{R}^d\times\mathbb{R}^d}\int_{\mathbb{R}^d\times\mathbb{R}^d} W\left(\frac{x-y}{\epsilon}\right) f_t(\mathrm{d}x,\mathrm{d}v) f_t(\mathrm{d}y,\mathrm{d}w),$$

whose critical points satisfy a nonlinear Vlasov system. When f_t is the push forward of a probability density ϱ_t defined on the physical space $\mathbb{R}^d$ by a map of the form $\mathbf{id} \times \mathbf{v}_t$ defined on $\mathbb{R}^d \times \mathbb{R}^d$ (where $\mathbf{id}$ is the identity map), the above actions are reduced to

$$\frac{1}{2}\int_0^T \int_{\mathbb{R}^d} |\mathbf{v}_t(x)|^2 \varrho_t(\mathrm{d}x) - \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} W\left(\frac{x-y}{\epsilon}\right)\varrho_t(\mathrm{d}x)\varrho_t(\mathrm{d}y). \tag{1.1}$$

In this work we study the Γ-limit of a class of functionals that includes those appearing in (1.1) and their link with the homogenization of systems of PDEs of the nonlinear Vlasov type. We endow $AC^p(0,T;\mathcal{P}_p(\mathbb{R}^d))$, the set of p-absolutely continuous paths on $\mathcal{P}_p(\mathbb{R}^d)$ [2], with a topology τ_w for which the sublevel sets of our actions are pre-compact (cf. Section 3). More precisely, τ_w is defined as follows: we say that $\{\sigma^n\}_n \subset AC^p(0,T;\mathcal{P}_p(\mathbb{R}^d))$ satisfying the condition

$$\sup_{n\in\mathbb{N}} \int_0^T \left(W_p^p(\sigma_t^n, \delta_0) + |(\sigma^n)'|_t^p\right)\mathrm{d}t < \infty$$

τ_w-converges to σ in $AC^p(0,T;\mathcal{P}_p(\mathbb{R}^d))$ if

$$\lim_{n\to\infty}\int_0^T W_1(\sigma_t^n,\sigma_t)\mathrm{d}t = 0.$$

Here, W_q is the L^q-Wasserstein distance for $1 \le q < \infty$ and $|\sigma'|$ is the metric derivative of $\sigma \in AC^p(0,T;\mathcal{P}_p(\mathbb{R}^d))$. For such paths σ and the associated (unique) velocity $\mathbf{v}$ of minimal $\mathbf{L}^p(\sigma)$-norm [2], we define

$$\mathcal{F}^\epsilon(\sigma) = \int_0^T \mathcal{L}(D_\#^{1/\epsilon}\sigma_t, \mathbf{v}_t \circ D^\epsilon)\mathrm{d}t, \tag{1.2}$$

where D^ϵ is the map of $\mathbb{R}^d$ onto itself defined by $D^\epsilon x = \epsilon x$. The notation $\mathbf{L}^p$ replaces L^p when dealing with $\mathbb{R}^d$-valued p-integrable functions. If $d = 1$, we stick with the classical notation L^p. We have denoted the push forward operator by $\# : \mathcal{P}_p(\mathbb{R}^d) \to \mathcal{P}_p(\mathbb{R}^d)$. The topology τ_w seems to be too weak to directly provide information on the $\Gamma(\tau_w)$-limit of the functionals $\mathcal{F}^\epsilon$. Our strategy is to introduce a stronger topology τ on a subset of $AC^p(0,T;\mathcal{P}_p(\mathbb{R}^d))$, hoping to be able to extract enough information from the $\Gamma(\tau)$-limit of the functionals $\mathcal{F}^\epsilon$ in order to draw some conclusions on the $\Gamma(\tau_w)$-limit. The strategy completely paid off when $d = 1$, and produced partial results in the multi-dimensional case. The τ-topology that we consider is inspired from the well-known isometry between the convex cone of nondecreasing functions