Available online at www.sciencedirect.com
SciVerse ScienceDirect
ADVANCES IN Mathematics
Advances in Mathematics 229 (2012) 731–769
www.elsevier.com/locate/aim

Compactness results for immersions of prescribed Gaussian curvature I – analytic aspects

Graham Smith

Departament de Matemàtiques, Facultat de Ciències, Edifici C, Universitat Autònoma de Barcelona, 08193 Bellaterra, Barcelona, Spain

Received 10 December 2009; accepted 17 October 2011

Available online 8 November 2011

Communicated by Luis Caffarelli

Dedicated to Hennie Smith
22/12/1914–08/12/2009
With affection

Abstract

We extend recent results of Guan and Spruck, proving existence results for constant Gaussian curvature hypersurfaces in Hadamard manifolds.

MSC: primary 58E12; secondary 35J25, 35J60, 53A10, 53C21, 53C42

Keywords: Gaussian curvature; Plateau problem; Monge–Ampère equation; Non-linear elliptic PDEs

1. Introduction

Let $M := M^{n+1}$ be an $(n+1)$-dimensional Riemannian manifold. An immersed hypersurface in M is a pair $(\Sigma, \partial\Sigma) := ((S, \partial S), i)$ where $(S, \partial S)$ is a compact, n-dimensional manifold with boundary and $i : S \to M$ is an immersion (that is, a smooth mapping whose derivative is everywhere injective). Throughout the sequel we abuse notation and denote $(S, \partial S)$ also by $(\Sigma, \partial\Sigma)$. We recall that the shape operator of the immersion is defined at each point by taking the covariant derivative in M of the unit, normal vector field over Σ at that point, and that the

E-mail address: grahamandrewsmith@gmail.com.

0001-8708/$ – see front matter
doi:10.1016/j.aim.2011.10.008

Gaussian curvature (also called the extrinsic curvature) is then defined as a function over S to be equal to the determinant of the shape operator at each point.

Geometers have studied the concept of Gaussian curvature ever since Gauss first proved in [6] his famous *Teorema Egregium* which states that the Gaussian curvature of a surface immersed in $\mathbb{R}^3$ only depends on its intrinsic geometry and not on the immersion, which explains, for example, why a flat sheet of paper cannot be smoothly wrapped round a portion of the sphere. In more recent times, the Gaussian curvature of a hypersurface has revealed itself as an interesting object of study also from the perspective of geometric analysis as a straightforward and archetypal case of a much larger class of problems, including those of affine geometry, mass transport, Calabi–Yau geometry and so on, all of whose underlying equations are of so-called Monge–Ampère type.

In studying hypersurfaces of constant curvature of any sort, the most natural problems to study are those of Plateau and Minkowski, which ask respectively for the existence of hypersurfaces of constant curvature with prescribed boundary, or without boundary but instead satisfying certain topological conditions. The study of these problems has enjoyed a rich development over the last century, with the application of a wide variety of different techniques, including, for example, polyhedral approximation, used by Pogorelov to solve the Minkowski problem for convex, immersed spheres of prescribed Gaussian curvature in Euclidean space (cf. [16]), and, more recently, the continuity method, as used by Caffarelli, Nirenberg and Spruck (cf. [3]) to solve the Plateau problem for locally strictly convex (LSC) hypersurfaces which are graphs over a given hyperplane in Euclidean space. The ideas of Caffarelli, Nirenberg and Spruck were further developed in one direction by Rosenberg and Spruck (cf. [17]) to prove the existence of LSC hypersurfaces of constant extrinsic curvature in hyperbolic space with prescribed asymptotic boundary in the sphere at infinity (which was in turn generalized by Guan and Spruck in [11] and [12] to treat more general notions of curvature). Likewise they were developed in another direction by Guan and Spruck in [9] to prove existence of LSC hypersurfaces of constant extrinsic curvature in Euclidean space with prescribed boundary in the unit sphere. This led Spruck to conjecture in [23] that any compact, codimension 2, immersed submanifold in Euclidean space which is the boundary of an LSC, immersed hypersurface is also the boundary of an LSC, immersed hypersurface of constant Gaussian curvature, a conjecture which was confirmed simultaneously by Guan and Spruck in [10] and Trudinger and Wang in [24] using in both cases a combination of Caffarelli, Nirenberg and Spruck's continuity method alongside an elegant application of the Perron method.

With the exception of [17], the above results essentially concern submanifolds of $\mathbb{R}^{n+1}$ and mild generalizations of this setup, and since most of the techniques used above rely in some way or another on the geometry of Euclidean space, the problem in general ambient manifolds has remained largely open. Nonetheless, in [14], Labourie showed how pseudo-holomorphic geometry may be applied in conjunction with a parametric version of the continuity method to solve the Plateau problem in the case where M is a 3-dimensional Hadamard manifold. However, since this approach relies on techniques of holomorphic function theory, it does not easily generalize to the higher dimensional case, which has therefore hitherto remained unsolved. It is to fill this gap that we present in this and our forthcoming work [18] an approach which allows us to solve the Plateau problem for hypersurfaces of constant (or prescribed) Gaussian curvature in general manifolds, thus generalizing the results [10] and [24] of Guan and Spruck and Trudinger and Wang on the one hand and the result [14] of Labourie on the other. In the interest of simplicity, we henceforth restrict attention to **Hadamard manifolds**, which, we recall, are, by definition, complete, simply connected manifolds of non-positive sectional curvature. We leave the enthusiastic reader to investigate the few extra technical conditions required to state and prove the results in general manifolds.