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On the obstructed Lagrangian Floer theory ☆

Cheol-Hyun Cho

Department of Mathematics and Research Institute of Mathematics, Seoul National University, Kwanakgu Shinrim, San 56-1 Seoul, South Korea

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Abstract

Lagrangian Floer homology in a general case has been constructed by Fukaya, Oh, Ohta and Ono, where they construct an A_∞-algebra or an A_∞-bimodule from Lagrangian submanifolds. They developed obstruction and deformation theories of the Lagrangian Floer homology theory. But for obstructed Lagrangian submanifolds, the standard Lagrangian Floer homology cannot be defined.

We explore several well-known homology theories on these A_∞-objects, which are Hochschild and cyclic homology for an A_∞-objects and Chevalley–Eilenberg or cyclic Chevalley–Eilenberg homology for their underlying L_∞-objects. We show that these homology theories are well-defined and invariant even in the obstructed cases. Due to the existence of m_0, the standard homological algebra does not work and we develop analogous homological algebra over Novikov fields.

We provide computations of these homology theories in some cases: We show that for an obstructed A_∞-algebra with a non-trivial primary obstruction, Chevalley–Eilenberg Floer homology vanishes, whose proof is inspired by the comparison with cluster homology theory of Lagrangian submanifolds by Cornea and Lalonde.

In contrast, we also provide an example of an obstructed case whose cyclic Floer homology is non-vanishing.

Keywords: Symplectic geometry; Lagrangian submanifold; Floer homology; Hochshild homology; Cyclic homology

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E-mail address: chocheol@snu.ac.kr.

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1. Introduction

Floer homology invented by Floer [10], has proven to be a very powerful tool in the symplectic geometry and related areas. For Floer cohomology of a Lagrangian submanifold in a symplectic manifold, Fukaya, Oh, Ohta and Ono [14] have constructed the A_∞-algebra of a Lagrangian submanifold and the A_∞-bimodule of a pair of Lagrangian submanifolds in full generality. An A_∞-algebra is given by sequence of operations m_k for $k = 0, 1, 2, \ldots,$ satisfying quadratic relations (see Definition 2.2). If $m_0 = 0$, the m_1 is a differential and the homology of m_1 becomes Lagrangian Floer cohomology for A_∞-algebra of a Lagrangian submanifold. But in general, $m_0 \neq 0$ and in such a case, one cannot consider m_1-homology. In [14], they have developed obstruction and deformation theory and showed that if the obstructions vanish, then one can deform the given A_∞-structure to m_k^b for $k = 0, 1, 2, \ldots,$ so that the deformed A_∞-algebra has vanishing m_0^b. In such unobstructed (or weakly obstructed) cases, the Floer cohomology theories can be defined, and can be applied to the study of symplectic topology or homological mirror symmetry (see [21,3,6,4,14,15] for example).

In this paper, we will investigate the obstructed cases, for which the standard Floer cohomology cannot be defined. We explore alternative ways of defining homology, by considering Hochschild and cyclic homology of A_∞-objects and Chevalley–Eilenberg or cyclic Chevalley–Eilenberg homology of their underlying L_∞-objects.

Such homology theories are well known for associative (or Lie) algebras (see for example [23]). Also for A_∞, or L_∞-algebras with $m_0 = 0$ such homology theories have been known (we refer readers to [19] for the definitions using non-commutative geometry). The definition easily extends to the case with non-vanishing m_0, but it turns out that the usual homological algebra does not immediately extend as the usual contracting homotopy of the bar complex does not work with $m_0 \neq 0$. We show that by working with Novikov fields, modified contraction homotopy exists and that we still have the reduced Hochschild homology, and (b, B)-cyclic complex where Connes–Tsygan B-operator actually has an additional term compared to the standard case.

The main motivation to study these homology theories is to have a well-defined Floer homology theory even for obstructed Lagrangian submanifolds. We show that even for obstructed A_∞-algebras, these homology theories are well-defined and invariant under various choices involved and hence define invariants of a homotopy class of A_∞-objects.

One can obtain cyclic homology complex of an A_∞-algebra or a cyclic Chevalley–Eilenberg homology complex from the bar complex. Recall that A_∞-algebra $(C, \{m_k\})$ is algebraically a tensor-coalgebra $T(C[1])$ with a codifferential $\widehat{d} = \sum_k \widehat{m}_k$. The complex $(T(C[1]), \widehat{d})$ is called a bar complex, whose homology is trivial (see Lemma 3.1). One can consider cyclic or symmetric bar complex, which is a subcomplex of the bar complex, by considering the fixed elements of the natural cyclic or symmetric group action. The homology of these subcomplexes is in fact the cyclic homology of A_∞-algebra or cyclic Chevalley–Eilenberg homology of the induced L_∞-algebra. Here, as any associative algebra can be regarded as a Lie algebra whose bracket is given by the commutator, an A_∞-algebra (A_∞-module) gives rise to an underlying L_∞-algebra (L_∞-module) by symmetrizing all A_∞-operations and Chevalley–Eilenberg homology is their Lie algebra homology. We remark that the existence and invariance of homology of these subcomplexes has been known to authors of [14]. (Note that what we call cyclic bar complex is different from the cyclic bar complex of Getzler and Jones [18].)

We also remark that there have been different approaches to consider obstructed cases, by Cornea and Lalonde [7] using Morse functions and by Fukaya [11] using the relationship with loop space homology and Floer homology.