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A Brylinski filtration for affine Kac–Moody algebras

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Abstract

Braverman and Finkelberg have recently proposed a conjectural analogue of the geometric Satake isomorphism for untwisted affine Kac–Moody groups. As part of their model, they conjecture that (at dominant weights) Lusztig's q-analog of weight multiplicity is equal to the Poincare series of the principal nilpotent filtration of the weight space, as occurs in the finite-dimensional case. We show that the conjectured equality holds for all affine Kac–Moody algebras if the principal nilpotent filtration is replaced by the principal Heisenberg filtration. The main body of the proof is a Lie algebra cohomology vanishing result. We also give an example to show that the Poincare series of the principal nilpotent filtration is not always equal to the q-analog of weight multiplicity. Finally, we give some partial results for indefinite Kac–Moody algebras.

Keywords: Affine Kac–Moody algebra; Representation theory; Brylinski filtration; Cohomology vanishing; Kostka–Foulkes polynomials

1. Introduction

Let $\mathcal{L}(\lambda)$ be an integrable highest-weight representation of a symmetrizable Kac–Moody algebra $\mathfrak{g}$. The Kostant partition functions $K(\beta; q)$ are defined for weights β by

$$\sum_{\beta} K(\beta; q)e^{\beta} = \prod_{\alpha \in \Delta^+} \left(1 - qe^{\alpha}\right)^{-\operatorname{mult}\alpha},$$

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where Δ^+ is the set of positive roots and $\operatorname{mult}\alpha = \dim \mathfrak{g}_\alpha$. The q-character of a weight space $\mathcal{L}(\lambda)_\mu$ is the function

$$m_\mu^\lambda(q) = \sum_{w \in W} \epsilon(w) K(w * \lambda - \mu; q), \tag{1}$$

where W is the Weyl group of $\mathfrak{g}$, ϵ is the usual sign representation of W, and $w * \lambda = w(\lambda + \rho) - \rho$ is the shifted action of W. The name "q-character" is used because $m_\mu^\lambda(1) = \dim \mathcal{L}(\lambda)_\mu$.

When $\mathfrak{g}$ is finite-dimensional it is well known that the q-analogs $m_\mu^\lambda(q)$ are equal to Kostka–Foulkes polynomials, which express the characters of highest-weight representations in terms of Hall–Littlewood polynomials [10,7], and are Kazhdan–Lusztig polynomials for the affine Weyl group [12]. When μ is dominant the coefficients of $m_\mu^\lambda(q)$ are non-negative. There is an explanation for this phenonemon, first conjectured by Lusztig [12]: the weight space $\mathcal{L}(\lambda)_\mu$ has an increasing filtration ${}^eF^*$ such that $m_\mu^\lambda(q)$ is equal to the Poincare polynomial

$${}^eP_\mu^\lambda(q) = \sum_{i \geqslant 0} q^i \dim {}^eF^i \mathcal{L}(\lambda)_\mu \,/\, {}^eF^{i-1} \mathcal{L}(\lambda)_\mu \tag{2}$$

of the associated graded space. This identity was first proved by Brylinski for μ regular or $\mathfrak{g}$ of classical type; the filtration ${}^eF^*$ is known as the *Brylinski* or *Brylinski–Kostant* filtration, and is defined by

$${}^eF^i\big(\mathcal{L}(\lambda)_\mu\big) = \big\{v \in \mathcal{L}(\lambda)_\mu \colon e^{i+1}v = 0\big\},$$

where e is a principal nilpotent. Brylinski's proof was extended to all dominant weights by Broer [3]. More recently Joseph, Letzter, and Zelikson gave a purely algebraic proof of the identity $m_\mu^\lambda = {}^eP_\mu^\lambda$, and determined ${}^eP_\mu^\lambda$ for μ non-dominant [8]. Viswanath has shown that the q-analogs of weight multiplicity of an arbitrary symmetrizable Kac–Moody are Kostka–Foulkes polynomials for generalized Hall–Littlewood polynomials, and determined $m_\mu^\lambda(q)$ at some simple μ for an untwisted affine Kac–Moody [15].

The point of this paper is to extend Brylinski's result to affine (i.e. indecomposable of affine type) Kac–Moody algebras. We show that, as in the finite-dimensional case, there is a filtration on $\mathcal{L}(\lambda)_\mu$ such that when μ is dominant, $m_\mu^\lambda(q)$ is equal to the Poincare series of the associated graded space. Unlike the finite-dimensional case, the principal nilpotent is not sufficient to define the filtration in the affine case; instead, we use the positive part of the principal Heisenberg (this form of Brylinski's identity was first conjectured by Teleman). Brylinski's original proof of the identity $m_\mu^\lambda = {}^eP_\mu^\lambda$ uses a cohomology vanishing result for the flag variety. Our proof is based on the same idea, but uses the Lie algebra cohomology approach of [6]. In particular we prove a vanishing result for Lie algebra cohomology by calculating the Laplacian with respect to a Kahler metric. Although we concentrate on the affine case for simplicity, our results generalize easily to the case when $\mathfrak{g}$ is a direct sum of algebras of finite or affine type. There are two difficulties in extending this result to indefinite symmetrizable Kac–Moody algebras: there does not seem to be a simple analogue of the Brylinski filtration, and the cohomology vanishing result does not extend for all dominant weights μ. We can overcome these difficulties by replacing the Brylinski filtration with an intermediate filtration, and by requiring that the root $\lambda - \mu$ has affine support. Thus we get some partial non-negativity results for the coefficients of $m_\mu^\lambda(q)$ even when $\mathfrak{g}$ is of indefinite type.