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Hamiltonian group actions on symplectic Deligne–Mumford stacks and toric orbifolds

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Abstract

We develop differential and symplectic geometry of differentiable Deligne–Mumford stacks (orbifolds) including Hamiltonian group actions and symplectic reduction. As an application we construct new examples of symplectic toric DM stacks.

Keywords: Deligne–Mumford stack; Symplectic geometry; Symplectic reduction; Toric

1. Introduction

We have three goals in this paper. The most fundamental is to write down in a consistent form the basics of differential and symplectic geometry of orbifolds thought of as Deligne–Mumford (DM) stacks over the category of smooth manifolds. This includes descriptions of the tangent and cotangent bundles, vector fields, differential forms, Lie group actions, and symplectic reduction. Most if not all of these notions are well-known on the level of being "analogous to manifolds". Recall that in the original approach of Satake [14] an orbifold is a topological space which is locally a quotient of a vector space by a finite group action. Smooth functions invariant under these local group actions form the structure sheaf. A more recent incarnation of this idea, largely due to Haefliger, is to think of an atlas on an orbifold as a proper étale Lie groupoid [11]. This approach makes it easy to define local geometric structures such as vector

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fields, differential forms, symplectic structures and Morse functions. However global structures such as Lie group actions are awkward to work with in an étale atlas. One of our observations is that global structures look much simpler in suitable non-étale atlases. So we prefer to think of orbifolds as a Deligne–Mumford (DM) stacks and compute in arbitrary atlases. The downside is that in an arbitrary groupoid atlas vector fields and differential forms look more complicated. We show that there are consistent descriptions of all such geometric structures on a DM stack. More specifically, given a DM stack $\mathcal{X}$ there is a presentation $X_1 \rightrightarrows X_0 \to \mathcal{X}$ of $\mathcal{X}$ by a Lie groupoid $X_1 \rightrightarrows X_0$ so that any geometric structure on $\mathcal{X}$ is given by a compatible pair of the corresponding structures on X_1 and X_0. For example a vector field (differential form, function) is a compatible pair of vector fields (differential forms, functions) on X_1 and X_0. Similarly given a Lie group action on $\mathcal{X}$ there is an atlas $X_0 \to \mathcal{X}$ so that the action can be described by a pair of *free* actions on X_1 and X_0. Such a presentation of a group action is useful even in the case of manifolds, where it can be thought of as a stacky version of replacing a G-manifold M with $EG \times_G M$. Consequently the quotient of $\mathcal{X}$ with respect to a G-action is represented by the quotients of X_1 and X_0. Similar statements hold for symplectic quotients, etc. A reader not comfortable with the abstract stack theory can safely take these pair-based descriptions as definitions. This is perfectly fine for applications, since the actual calculations are always done in atlases. However to show that the definitions make sense one should either prove that they are atlas-independent (i.e., Morita-invariant) or convince oneself that there is an abstract definition in terms of the stack $\mathcal{X}$, the approach taken in this paper.

The paper is organized as follows. In Section 2 we discuss vector fields and forms on DM stacks and provide their description in non-étale atlases. We end the section with a definition of a symplectic DM stack.

In Section 3 we review group actions on stacks following Romagny [13], define Hamiltonian actions and prove an analogue of Marsden–Weinstein–Meyer reduction theorem for DM stacks.

In Section 4 we relate group actions on quotient stacks to group extensions. We then describe its symplectic analogue, which may be thought of as the stacky version of reduction in stages.

In Section 5 we take up symplectic toric DM stacks. Recall that symplectic toric manifolds are analogues of toric varieties in algebraic geometry, though symplectic-algebraic correspondence is not 1–1. Compact connected symplectic toric manifolds were classified by Delzant [4]. Delzant's classification was extended to compact orbifolds by Lerman and Tolman [9]. However the class of orbifolds is not as natural as the class of DM stacks. For example it is not closed under taking substacks. For this reason we feel it is preferable to work with symplectic toric DM stacks rather than orbifolds.

In algebraic geometry the corresponding notion of a toric DM stack is still evolving. To the best of our knowledge, it first appeared in the work of Borisov, Chen, and Smith [2] as a construction. Later Iwanari [8] proposed the definition of a toric triple as an effective DM stack with an action of an algebraic torus having a dense orbit isomorphic to the torus. Recently, Fantechi, Mann and Nironi [5] gave a new definition of a smooth toric DM stack as DM stack with an action of a DM torus $\mathcal{T}$ having a dense open orbit isomorphic to $\mathcal{T}$. According to [5], a DM torus is a Picard stack isomorphic to $T \times B\Gamma$ where T is an algebraic torus and $B\Gamma$ is the classifying stack of a finite *abelian* group Γ.

We define a symplectic toric DM stack as a symplectic DM stack with an effective Hamiltonian action of a compact torus. Then, generalizing a construction in [4], we produce a large class of examples of symplectic toric DM stacks as symplectic quotients of the form $(\mathbb{C}^N \times B\Gamma)/\!\!/_c A$, where Γ is an *arbitrary* finite group and A is a closed subgroup of $\mathbb{R}^N/\mathbb{Z}^N$ (symplectic quotients of DM stacks are defined in Theorem 3.13 below). From the point of view of symplectic