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Symmetric quivers, invariant theory, and saturation theorems for the classical groups

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Abstract

Let G denote either a special orthogonal group or a symplectic group defined over the complex numbers. We prove the following saturation result for G: given dominant weights $\lambda^1, \dots, \lambda^r$ such that the tensor product $V_{N\lambda^1} \otimes \cdots \otimes V_{N\lambda^r}$ contains nonzero G-invariants for some $N \geqslant 1$, we show that the tensor product $V_{2\lambda^1} \otimes \cdots \otimes V_{2\lambda^r}$ also contains nonzero G-invariants. This extends results of Kapovich–Millson and Belkale–Kumar and complements similar results for the general linear group due to Knutson–Tao and Derksen–Weyman. Our techniques involve the invariant theory of quivers equipped with an involution and the generic representation theory of certain quivers with relations.

Keywords: Quivers; Semi-invariants; Tensor product multiplicities; Classical groups; Invariant theory; Algebras with involution

1. Introduction

Throughout, we fix an algebraically closed field K. We shall assume that K is of characteristic 0 in the introduction. However, as some results in this paper extend to positive characteristic, we will mention what assumptions we make on the characteristic within each section of the paper.

When G is a reductive group defined over K, and λ is a dominant weight of G, the notation V_λ denotes an irreducible representation of G with highest weight λ. Also, if W is a representation

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of G, we use W^G to denote the subspace of G-invariants. The following theorem is the main result of the paper.

Theorem 1.1. *Let G be either a special orthogonal or symplectic group, and let $\lambda^1, \dots, \lambda^r$ be dominant weights of G. If $(V_{N\lambda^1} \otimes \cdots \otimes V_{N\lambda^r})^G \neq 0$ for some $N \geqslant 1$, then $(V_{2\lambda^1} \otimes \cdots \otimes V_{2\lambda^r})^G \neq 0$.*

We say that 2 is a **saturation factor** for the special orthogonal and symplectic groups. We will use the notation $\mathbf{SO}(m)$, $\mathbf{O}(m)$, and $\mathbf{Sp}(2n)$ to mean special orthogonal, orthogonal, and symplectic groups, respectively.

Corollary 1.2. *Let G be the spin group $\mathbf{Spin}(m)$, and let $\lambda^1, \dots, \lambda^r$ be dominant weights of G. If $(V_{N\lambda^1} \otimes \cdots \otimes V_{N\lambda^r})^G \neq 0$ for some $N \geqslant 1$, then $(V_{4\lambda^1} \otimes \cdots \otimes V_{4\lambda^r})^G \neq 0$.*

Proof. If λ is a dominant weight of $\mathbf{Spin}(m)$, then 2λ is a dominant weight of $\mathbf{SO}(m)$, and the action of $\mathbf{Spin}(m)$ factors through $\mathbf{SO}(m)$ on each $V_{4\lambda^i}$. □

1.1. History and related results

Before we give an outline for the proof of Theorem 1.1, we mention some historical context for the theorem and some results that have previously been proven in this direction.

The results start with the so-called saturation conjecture proven by Knutson and Tao [12] and Derksen and Weyman [6].

Theorem *(Knutson–Tao, Derksen–Weyman). Let $\lambda^1, \dots, \lambda^r$ be dominant weights of $G = \mathbf{GL}(n)$. If $(V_{N\lambda^1} \otimes \cdots \otimes V_{N\lambda^r})^G \neq 0$, then $(V_{\lambda^1} \otimes \cdots \otimes V_{\lambda^r})^G \neq 0$.*

This problem itself was inspired by Klyachko's solution [11] of Horn's problem of characterizing the possible eigenvalues of Hermitian matrices $A_1, \dots, A_r$ whose sum is 0. We leave the details out and refer to Fulton's paper [8] for a survey and further references.

When $r = 3$, this theorem can be restated in terms of the Littlewood–Richardson rule (see [21, Theorem 2.3.4] or [17, §12.5]), which gives an explicit combinatorial recipe for calculating the dimension of the G-invariant subspace of a triple tensor product, or equivalently, for calculating tensor product multiplicities. However, the formulation of this rule is not conducive to proving the saturation property. The proof of Knutson and Tao involves formulating a new combinatorial rule which more manifestly possesses the saturation property. However, this approach seems to be difficult to generalize. Our paper will follow the ideas of Derksen and Weyman. Before reviewing the ideas from that paper, we mention some other saturation results to put Theorem 1.1 into perspective. We refer the reader to [13] for more results and conjectures related to tensor product multiplicities.

Theorem *(Kapovich–Millson). Let G be a simple connected group over K, and let $\lambda^1, \dots, \lambda^r$ be dominant weights of G such that $\lambda^1 + \cdots + \lambda^r$ is in the root lattice of G. Let k be the least common multiple of the coefficients of the highest root of G written in terms of simple roots. If $(V_{N\lambda^1} \otimes \cdots \otimes V_{N\lambda^r})^G \neq 0$ for some $N \geqslant 1$, then $(V_{k^2\lambda^1} \otimes \cdots \otimes V_{k^2\lambda^r})^G \neq 0$.*

See [10, Corollary 7.3 and Remark 7.2]. For the special orthogonal and symplectic groups, this gives a saturation factor of 4, which our Theorem 1.1 improves to 2 (and drops the assumption