Available online at www.sciencedirect.com

SciVerse ScienceDirect

Advances in Mathematics 229 (2012) 1266–1299

ADVANCES IN Mathematics

www.elsevier.com/locate/aim

Regularity and decay of solutions of nonlinear harmonic oscillators

Marco Cappiello [a,*], Fabio Nicola [b]

[a] Dipartimento di Matematica, Università degli Studi di Torino, Via Carlo Alberto 10, 10123 Torino, Italy
[b] Dipartimento di Matematica, Politecnico di Torino, Corso Duca degli Abruzzi 24, 10129 Torino, Italy

Received 11 April 2011; accepted 28 October 2011

Available online 17 November 2011

Communicated by Charles Fefferman

Abstract

We prove sharp analytic regularity and decay at infinity of solutions of variable coefficients nonlinear harmonic oscillators. Namely, we show holomorphic extension to a sector in the complex domain, with a corresponding Gaussian decay, according to the basic properties of the Hermite functions in $\mathbb{R}^d$. Our results apply, in particular, to nonlinear eigenvalue problems for the harmonic oscillator associated to a real-analytic scattering, or asymptotically conic, metric in $\mathbb{R}^d$, as well as to certain perturbations of the classical harmonic oscillator.

MSC: 35J61; 35B65; 35B40; 35S05; 35A20

Keywords: Nonlinear harmonic oscillators; Holomorphic extension; Gaussian decay; Pseudodifferential operators

1. Introduction

The harmonic oscillator $H = -\Delta + |x|^2$ in $\mathbb{R}^d$ represents one of the simplest and yet more useful models for several physical phenomena, and its relevance both in Mathematical Analysis and Physics is well known. Its eigenfunctions, namely the Hermite functions $h_\alpha(x)$, are given by the formulae $h_\alpha(x) = p_\alpha(x)e^{-|x|^2/2}$, $\alpha \in \mathbb{N}^d$, where p_α is a polynomial of degree $|\alpha|$ (see

* Corresponding author.
E-mail addresses: marco.cappiello@unito.it (M. Cappiello), fabio.nicola@polito.it (F. Nicola).

0001-8708/$ – see front matter
doi:10.1016/j.aim.2011.10.018

e.g. [29]). Two remarkable features of the Hermite functions are their Gaussian decay at infinity, and their very high regularity. In fact, we have

$$\left|h_\alpha(x)\right| \lesssim e^{-c|x|^2} \quad \text{for } x \in \mathbb{R}^d, \qquad \left|\widehat{h_\alpha}(\xi)\right| \lesssim e^{-c|\xi|^2} \quad \text{for } \xi \in \mathbb{R}^d \tag{1.1}$$

for every $c < 1/2$, where $\widehat{h_\alpha}(\xi)$ denotes the Fourier transform of h_α. The functions h_α in fact extend to entire functions $h_\alpha(x+iy)$ in the complex space $\mathbb{C}^d$ and, for every $0 < \varepsilon < 1$, we have the estimates

$$\left|h_\alpha(x+iy)\right| \lesssim e^{-c|x|^2} \quad \text{in the sector } |y| < \varepsilon\big(1+|x|\big), \tag{1.2}$$

for some $c > 0$.

In this paper we wonder to what extent these properties continue to hold for *nonlinear* perturbations of the harmonic oscillator, possibly with variable coefficients. Relevant models are equations of the type

$$-\Delta u + |x|^2 u - \lambda u = F[u], \quad \lambda \in \mathbb{C}, \tag{1.3}$$

with a nonlinearity of the form $F[u] = \sum_{|\alpha|+|\beta|\leqslant 1} c_{\alpha\beta} x^\beta \partial^\alpha u^k$, $k \geqslant 2$. Cappiello, Gramchev and Rodino in [8] showed by a counterexample that generally, even in dimension $d = 1$, there can exist Schwartz solutions of (1.3) which do not extend to entire functions in $\mathbb{C}$. In fact, a refinement of their argument (see Section 5 below) shows that a sequence of complex singularities may occur, approaching a straight line at infinity. On the other hand, as a positive result, it was proved in [8] that every solution $u \in H^s(\mathbb{R}^d)$, $s > d/2+1$, of (1.3) extends to a holomorphic function $u(x+iy)$ on the strip $\{z \in \mathbb{C}^d\colon |\mathrm{Im}\, z| < T\}$ and satisfies there an estimate of the type $|u(x+iy)| \leqslant Ce^{-c|x|^2}$, for some $c, C, T > 0$. Similar results, namely, holomorphic extension to a *strip* and super-exponential decay, were proved in [8,11] for more general classes of elliptic operators with polynomial coefficients.

The above mentioned negative result as well as the estimates (1.2), valid in a sector in the linear case, suggest the possibility, even in the presence of certain nonlinear perturbations, of a holomorphic extension of the solutions to a *sector*, rather than only a strip, with a corresponding Gaussian decay estimate. In this paper we show, for a large class of equations including (1.3), even with *non-polynomial* coefficients, that this is in fact the case. The techniques developed here actually will apply to much more general differential (and pseudodifferential) operators. To motivate the class of operators we will consider, we first discuss a special yet important example.

Consider the equation $Pu = F[u]$, with

$$P = \sum_{j,k=1}^{d} g^{jk}(x)\partial_j\partial_k + \sum_{k=1}^{d} b_k(x)\partial_k + V(x), \tag{1.4}$$

where the functions g^{jk}, b_k, and the potential V are real-analytic in $\mathbb{R}^d$, and satisfy the following conditions.

We suppose that the matrix (g^{jk}) is real and symmetric and that there exists a constant $C > 0$ such that

$$\sum_{j,k=1}^{d} g^{jk}(x)\xi_j\xi_k \geqslant C^{-1}|\xi|^2 \quad \forall x, \xi \in \mathbb{R}^d, \tag{1.5}$$