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Oscillation in the initial segment complexity of random reals ☆

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Abstract

We study oscillation in the prefix-free complexity of initial segments of 1-random reals. For upward oscillations, we prove that $\sum_{n\in\omega} 2^{-g(n)}$ diverges iff $(\exists^\infty n) K(X \upharpoonright n) > n + g(n)$ for every 1-random $X \in 2^\omega$. For downward oscillations, we characterize the functions g such that $(\exists^\infty n) K(X \upharpoonright n) < n + g(n)$ for almost every $X \in 2^\omega$. The proof of this result uses an improvement of Chaitin's counting theorem—we give a tight upper bound on the number of strings $\sigma \in 2^n$ such that $K(\sigma) < n + K(n) - m$.

The work on upward oscillations has applications to the K-degrees. Write $X \leqslant_K Y$ to mean that $K(X \upharpoonright n) \leqslant K(Y \upharpoonright n) + O(1)$. The induced structure is called the K-degrees. We prove that there are comparable (Δ^0_2) 1-random K-degrees. We also prove that every lower cone and some upper cones in the 1-random K-degrees have size continuum.

Finally, we show that it is independent of ZFC, even assuming that the Continuum Hypothesis fails, whether all chains of 1-random K-degrees of size less than $2^{\aleph_0}$ have a lower bound in the 1-random K-degrees.

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"*Although this oscillatory behaviour is usually considered to be a nasty feature, we believe that it illustrates one of the great advantages of complexity: the possibility to study degrees of randomness.*"

Michiel van Lambalgen, PhD Dissertation [31, p. 145].

1. Introduction

We study both the height and depth of oscillations in the prefix-free complexity of initial segments of random reals. By definition, X is 1-random if and only if $K(X \upharpoonright n) \geqslant n - O(1)$.[1] On the other hand, $K(\sigma) \leqslant |\sigma| + K(|\sigma|) + O(1)$ for any string $\sigma \in 2^{<\omega}$ [5]. Hence $K(X \upharpoonright n) \leqslant n + K(n) + O(1)$. How does $K(X \upharpoonright n)$ behave between these bounds? This is the subject of the present paper and, from a different perspective, of our companion paper [25]. Our results have many forerunners in the literature; we mention the most relevant ones below.

First note that there is a subtle difference in the nature of the upper and lower bounds on $K(X \upharpoonright n)$. The constant in the lower bound depends in an essential way on X, unlike the constant in the upper bound. More substantially, though neither the lower nor the upper bound can be improved (if they are to hold for *all* 1-random X), they are not tight in quite the same sense. Solovay [30] showed that almost all reals infinitely often achieve the upper bound, i.e., $\liminf_{n\to\infty} n + K(n) - K(X \upharpoonright n)$ is finite for almost all $X \in 2^\omega$ (see [33]). This is *not* true of all 1-random reals, and in fact, it turns out to be a characterization of 2-randomness [24]. To see that the upper bound cannot be improved at all, note that a straightforward modification of Solovay's proof shows that if $S \subseteq \omega$ is infinite, then almost all reals infinitely often achieve the upper bound *on* S. On the other hand, Chaitin proved that *no* 1-random can infinitely often achieve the lower bound: if $X \in 2^\omega$ is 1-random, then $\liminf_{n\to\infty} K(X \upharpoonright n) - n = \infty$. This does not mean that the lower bound can be improved. In Corollary 3.2, we show that if $h : \omega \to \omega$ is unbounded, then there is a 1-random $X \in 2^\omega$ such that $(\exists^\infty n) K(X \upharpoonright n) < n + h(n)$.

If $X \in 2^\omega$ is 1-random, it cannot be the case that $K(X \upharpoonright n)$ stays close to either bound; instead it oscillates, sometimes being "close" to the upper bound and sometimes being "close" to the lower bound. This behavior was first explored by Solovay [30]. In Section 3 we examine upward oscillations, starting from a characterization of 1-randomness proved by the authors [25].

Ample Excess Lemma. *$X \in 2^\omega$ is 1-random iff* $\sum_{n\in\omega} 2^{n-K(X\upharpoonright n)} < \infty$.

Note that this strengthens Chaitin's result: if $X \in 2^\omega$ is 1-random, then not only does $K(X \upharpoonright n) - n$ tend to infinity, but it does so fast enough to make the series converge. An immediate consequence is that if $\sum_{n\in\omega} 2^{-g(n)}$ diverges, then $(\exists^\infty n) K(X \upharpoonright n) > n + g(n)$ for every 1-random $X \in 2^\omega$. This generalizes a result of Solovay, who assumed additionally that g was computable. Furthermore, this result is *tight*. We prove that if $\sum_{n\in\omega} 2^{-g(n)} < \infty$, then there is

[1] Here K denotes *prefix-free complexity*. See Section 2 for a review of the definitions, notation and results used in this paper, with an emphasis on effective randomness.