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On the Beurling dimension of exponential frames

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Abstract

We study Fourier frames of exponentials on fractal measures associated with a class of affine iterated function systems. We prove that, under a mild technical condition, the Beurling dimension of a Fourier frame coincides with the Hausdorff dimension of the fractal.
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Contents

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1. Introduction

A family of vectors $(e_i)_{i\in I}$ in a Hilbert space $\mathcal{H}$ is called a *frame* if there exist $m, M > 0$ such that

$$m\|f\|^2 \leqslant \sum_{i\in I}\left|\langle f, e_i\rangle\right|^2 \leqslant M\|f\|^2.$$

The constants m and M are called *lower and upper bounds* of the frame. If only the upper bound holds, then $(e_i)_{i\in I}$ is called a *Bessel sequence*, and the upper bound is called the *Bessel bound*.

Frames provide robust, basis-like (but generally nonunique) representations of vectors in a Hilbert space. The potential redundancy of frames often allows them to be more easily constructible than bases, and to possess better properties than those that are achievable using bases. For example, redundant frames offer more resilience to the effects of noise or to erasures of frame elements than bases. Frames were introduced by Duffin and Schaeffer [2] in the context of nonharmonic Fourier series, and today they have applications in a wide range of areas. Following Duffin and Schaeffer a *Fourier frame* or *frame of exponentials* is a frame of the form $\{e^{2\pi i\lambda\cdot x}\}_{\lambda\in\Lambda}$ for the Hilbert space $L^2[0,1]$. Fourier frames are also closely connected with sampling sequences or complete interpolating sequences [17].

The main result of Duffin and Schaeffer is a sufficient density condition for $\{e^{2\pi i\lambda\cdot x}\}_{\lambda\in\Lambda}$ to be a frame. Landau [15], Jaffard [11] and Seip [19] "almost" characterize the frame properties of $\{e^{2\pi i\lambda\cdot x}\}_{\Lambda\in\Lambda}$ in terms of lower Beurling density:

$$\mathcal{D}^-(\Lambda) := \liminf_{h\to\infty}\inf_{x\in\mathbb{R}}\frac{\#(\Lambda\cap[x-h,x+h])}{2h}.$$

Theorem 1.1. *For $\{e^{2\pi i\lambda\cdot x}\}_{\Lambda\in\Lambda}$ to be a frame for $L^2[0,1]$, it is necessary that Λ is relatively separated and $\mathcal{D}^-(\Lambda)\geqslant 1$, and it is sufficient that Λ is relatively separated and $\mathcal{D}^-(\Lambda) > 1$.*

The property of relative separation is equivalent to the condition that the upper density

$$\mathcal{D}^+(\Lambda) := \limsup_{h\to\infty}\sup_{x\in R}\frac{\#(\Lambda\cap[x-h,x+h])}{2h}$$

is finite.

For the critical case when $\mathcal{D}^-(\Lambda) = 1$, the complete characterization was beautifully formulated by Joaquim Ortega-Cerdà and Kristian Seip in [17] where the key step is to connect the problem with de Branges' theory of Hilbert spaces of entire functions, and this new characterization lead to applications in a classical inequality of H. Landau and an approximation problem for subharmonic functions.

In recent years there has been a wide range of interests in expanding the classical Fourier analysis to fractal or more general probability measures [3,8,10,13,12,14,16,20–22]. One of the central themes of this area of research involves constructive and computational bases in $L^2(\mu)$-Hilbert spaces, where μ is a measure which is determined by some self-similarity property. Hence these include classical Fourier bases, as well as wavelet and frame constructions.