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Classifying spaces for braided monoidal categories and lax diagrams of bicategories ☆

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Abstract

This work contributes to clarifying several relationships between certain higher categorical structures and the homotopy type of their classifying spaces. Bicategories (in particular monoidal categories) have well-understood simple geometric realizations, and we here deal with homotopy types represented by lax diagrams of bicategories, that is, lax functors to the tricategory of bicategories. In this paper, it is proven that, when a certain bicategorical Grothendieck construction is performed on a lax diagram of bicategories, then the classifying space of the resulting bicategory can be thought of as the homotopy colimit of the classifying spaces of the bicategories that arise from the initial input data given by the lax diagram. This result is applied to produce bicategories whose classifying space has a double loop space with the same homotopy type, up to group completion, as the underlying category of any given (non-necessarily strict) braided monoidal category. Specifically, it is proven that these double delooping spaces, for categories enriched with a braided monoidal structure, can be explicitly realized by means of certain genuine simplicial sets characteristically associated to any braided monoidal categories, which we refer to as their (Street's) geometric nerves.

MSC: 18D05; 18D10; 55P15; 55P48

Keywords: Bicategory; Tricategory; Monoidal category; Braided monoidal category; Grothendieck construction; Nerve; Classifying space; Homotopy type; Loop space

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1. Introduction and summary

Higher-dimensional categories provide a suitable setting for the treatment of an extensive list of subjects with recognized mathematical interest. The construction of nerves and classifying spaces of higher categorical structures reveals ways to transport categorical coherence to homotopical coherence and it has shown its relevance as a tool in algebraic topology, algebraic geometry, algebraic K-theory, string field theory, conformal field theory, and in the study of geometric structures on low-dimensional manifolds. In particular, *braided monoidal categories* [24] have been playing a key role in recent developments in quantum theory and its related topics, mainly thanks to the following result, which was the starting point for this paper:

> *"The group completion of the classifying space of a braided monoidal category is a double loop space"*

as was noticed by J.D. Stasheff in [34], but originally proven by Z. Fiedorowicz in [13, Theorem 2] (some other proofs can be found in [4, Theorem 1.2] or in [2, Theorem 2.2], for example). More precisely, given any braided monoidal category

$$(\mathcal{M},\otimes,\boldsymbol{c})=(\mathcal{M},\otimes,\mathrm{I},\boldsymbol{a},\boldsymbol{l},\boldsymbol{r},\boldsymbol{c}),$$

Stasheff–Fiedorowicz's theorem implies the existence of a path-connected, simply connected space, uniquely defined up to homotopy equivalence, $\mathrm{B}(\mathcal{M},\otimes,\boldsymbol{c})$, and a homotopy-natural map $\mathrm{B}\mathcal{M}\to\Omega^2\mathrm{B}(\mathcal{M},\otimes,\boldsymbol{c})$, where $\mathrm{B}\mathcal{M}$ is the classifying space of the underlying category $\mathcal{M}$, which is, up to group completion, a homotopy equivalence. Hereafter, we shall refer to $\mathrm{B}(\mathcal{M},\otimes,\boldsymbol{c})$ both as the *classifying space of the braided monoidal category* and as the *double delooping* of $\mathrm{B}\mathcal{M}$, induced by the braided monoidal structure given on $\mathcal{M}$.

However, there is a problem with the space $\mathrm{B}(\mathcal{M},\otimes,\boldsymbol{c})$ since its existence is proven as an application of May's theory of E_2-operads [29] and, therefore, its various known constructions are based on some complicated and irritating processes of rectifying homotopy coherent diagrams. In fact, the double delooping construction is provided by May's bar-construction that only takes place after replacing $(\mathcal{M},\otimes,\boldsymbol{c})$ by an equivalent *strict* braided monoidal category $(\mathcal{M}',\otimes',\boldsymbol{c}')$, and then by carrying out a substitution of $\mathrm{B}\mathcal{M}'$ by a homotopy equivalent space upon which the little square operad of Boardman–Vogt acts [5], which depends on an explicit equivalence of operads between the braided operad used and the little 2-cube one. The resulting CW-complex thus obtained has many cells with little apparent intuitive connection with the data of the original monoidal category, and this leads one to search for any simplicial set, say "nerve of the braided monoidal category", realizing the space $\mathrm{B}(\mathcal{M},\otimes,\boldsymbol{c})$ and whose cells give a logical geometric meaning to the data of the braided monoidal category.

A natural response for that nerve was postulated in the nineties by J. Dolan and R. Street (probably among others) and it is as follows: since a braided monoidal category can be regarded as a one-object, one-arrow tricategory [17, Corollary 8.7] and each category as a tricategory whose 2-cells and 3-cells are all identities, one can consider strictly unitary lax functors from the categories $[p]=\{0<1<\cdots<p\}$ to the tricategory $\Omega^{-2}\mathcal{M}$ that the braided monoidal category $(\mathcal{M},\otimes,\boldsymbol{c})$ defines. Then, its *geometric nerve* is the simplicial set

$$Z^3(\mathcal{M},\otimes,\boldsymbol{c}):[p]\mapsto\mathrm{NorLaxFunc}\big([p],\Omega^{-2}\mathcal{M}\big),$$