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Advances in Mathematics 226 (2011) 4492–4532

ADVANCES IN Mathematics

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Skew quasisymmetric Schur functions and noncommutative Schur functions ☆

C. Bessenrodt [a], K. Luoto [b], S. van Willigenburg [b,*]

[a] *Institut für Algebra, Zahlentheorie und Diskrete Mathematik, Leibniz Universität, D-30167 Hannover, Germany*
[b] *Department of Mathematics, University of British Columbia, Vancouver, BC V6T 1Z2, Canada*

Received 6 August 2010; accepted 15 December 2010

Available online 11 January 2011

Communicated by Ezra Miller

Abstract

Recently a new basis for the Hopf algebra of quasisymmetric functions *QSym*, called quasisymmetric Schur functions, has been introduced by Haglund, Luoto, Mason, van Willigenburg. In this paper we extend the definition of quasisymmetric Schur functions to introduce skew quasisymmetric Schur functions. These functions include both classical skew Schur functions and quasisymmetric Schur functions as examples, and give rise to a new poset $\mathcal{L}_C$ that is analogous to Young's lattice. We also introduce a new basis for the Hopf algebra of noncommutative symmetric functions *NSym*. This basis of *NSym* is dual to the basis of quasisymmetric Schur functions and its elements are the pre-image of the Schur functions under the forgetful map $\chi : \mathit{NSym} \to \mathit{Sym}$. We prove that the multiplicative structure constants of the noncommutative Schur functions, equivalently the coefficients of the skew quasisymmetric Schur functions when expanded in the quasisymmetric Schur basis, are nonnegative integers, satisfying a Littlewood–Richardson rule analogue that reduces to the classical Littlewood–Richardson rule under χ.

As an application we show that the morphism of algebras from the algebra of Poirier–Reutenauer to *Sym* factors through *NSym*. We also extend the definition of Schur functions in noncommuting variables of Rosas–Sagan in the algebra *NCSym* to define quasisymmetric Schur functions in the algebra *NCQSym*. We prove these latter functions refine the former and their properties, and project onto quasisymmetric Schur functions under the forgetful map. Lastly, we show that by suitably labeling $\mathcal{L}_C$, skew quasisymmetric Schur functions arise in the theory of Pieri operators on posets.

☆ The second and third authors were supported in part by the National Sciences and Engineering Research Council of Canada. The third author was supported in part by the Alexander von Humboldt Foundation.

* Corresponding author.

E-mail addresses: bessen@math.uni-hannover.de (C. Bessenrodt), kwluoto@math.ubc.ca (K. Luoto), steph@math.ubc.ca (S. van Willigenburg).

0001-8708/$ – see front matter
doi:10.1016/j.aim.2010.12.015

MSC: primary 05E05, 05E15; secondary 05A05, 06A07, 16T05, 20C30

Keywords: Composition; Coproduct; Free Schur function; Skew Schur function; Littlewood–Richardson rule; Noncommutative symmetric function; *NCSym*; *NCQSym*; Pieri rule; Pieri operator; Poset; Quasisymmetric function; Symmetric function; Tableau

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1. Introduction

At the beginning of the last century, Schur [58] identified functions that would later bear his name as characters of the irreducible polynomial representations of $GL(n,\mathbb{C})$. These functions subsequently rose further in importance due to their ubiquitous nature. For example, in combinatorics they are the generating functions for semistandard Young tableaux, while in the representation theory of the symmetric group they form the image of the irreducible characters under the characteristic map. However, one of their most significant impacts has been as an orthonormal basis for the graded Hopf algebra of symmetric functions, *Sym*. More precisely, given partitions λ, μ, the expansion of the product of Schur functions s_λ, s_μ in this basis is

$$s_\lambda s_\mu = \sum_\nu c_{\lambda\mu}^\nu s_\nu,$$

where the $c_{\lambda\mu}^\nu$ are known as Littlewood–Richardson coefficients. However, this is not the only instance of Littlewood–Richardson coefficients. In the ordinary representation theory of the symmetric group, taking the induced tensor product of Specht modules S^λ and S^μ results in

$$\left(S^\lambda \otimes S^\mu\right)\uparrow^{S_n} = \bigoplus_\nu c_{\lambda\mu}^\nu S^\nu.$$