Available online at www.sciencedirect.com
ScienceDirect
Bull. Sci. math. 138 (2014) 705–720
BULLETIN DES SCIENCES MATHÉMATIQUES
www.elsevier.com/locate/bulsci

On maximal commutative subalgebras of Poisson algebras associated with involutions of semisimple Lie algebras

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Received 1 November 2013
Available online 4 December 2013

Abstract

For any involution σ of a semisimple Lie algebra $\mathfrak{g}$, one constructs a non-reductive Lie algebra $\mathfrak{k}$, which is called a $\mathbb{Z}_2$-contraction of $\mathfrak{g}$. In this paper, we attack the problem of describing maximal commutative subalgebras of the Poisson algebra $\mathcal{S}(\mathfrak{k})$. This is closely related to the study of the coadjoint representation of $\mathfrak{k}$ and the set, $\mathfrak{k}^*_{reg}$, of the regular elements of $\mathfrak{k}^*$. By our previous results, in the context of $\mathbb{Z}_2$-contractions, the argument shift method provides maximal commutative subalgebras of $\mathcal{S}(\mathfrak{k})$ whenever $\operatorname{codim}(\mathfrak{k}^* \setminus \mathfrak{k}^*_{reg}) \geqslant 3$.

Our main result here is that $\operatorname{codim}(\mathfrak{k}^* \setminus \mathfrak{k}^*_{reg}) \geqslant 3$ if and only if the Satake diagram of σ has no trivial nodes. (A node is trivial, if it is white, has no arrows attached, and all adjacent nodes are also white.) The list of suitable involutions is provided. We also describe certain maximal commutative subalgebras of $\mathcal{S}(\mathfrak{k})$ if the (-1)-eigenspace of σ in $\mathfrak{g}$ contains regular elements.

MSC: 17B63; 17B70; 14L30; 17B08; 13A50

Keywords: Lie–Poisson bracket; Symmetric pair; Isotropy representation; Argument shift; $\mathbb{Z}_2$-contraction

0. Introduction

Let Q be a connected algebraic group, with Lie algebra $\mathfrak{q}$, over an algebraically closed field $\Bbbk$ of characteristic zero. The symmetric algebra $\mathcal{S}(\mathfrak{q}) \simeq \Bbbk[\mathfrak{q}^*]$ is equipped with the standard

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0007-4497/$ – see front matter
http://dx.doi.org/10.1016/j.bulsci.2013.11.004

Lie–Poisson bracket { , }, and the algebra of invariants $\mathcal{S}(\mathfrak{q})^Q$ is the centre of $(\mathcal{S}(\mathfrak{q}), \{\ ,\ \})$. We say that a subalgebra $\mathcal{A} \subset \mathcal{S}(\mathfrak{q})$ is *commutative* if the bracket { , } vanishes on $\mathcal{A}$. As is well known, a commutative subalgebra cannot have the transcendence degree larger than $(\dim\mathfrak{q} + \operatorname{ind}\mathfrak{q})/2$, where $\operatorname{ind}\mathfrak{q}$ is the index of $\mathfrak{q}$. If this bound is attained, then $\mathcal{A}$ is said to be *of maximal dimension*. A commutative subalgebra $\mathcal{A}$ is said to be *maximal*, if it is not contained in a larger commutative subalgebra of $\mathcal{S}(\mathfrak{q})$. It is shown in [9] that natural commutative subalgebras of $\mathcal{S}(\mathfrak{q})$ can be constructed through the use of $\mathcal{S}(\mathfrak{q})^Q$ and any $\xi \in \mathfrak{q}^*$. This procedure is known as the "argument shift method", see Section 1.2 for details. We write $\mathcal{F}_\xi(\mathcal{S}(\mathfrak{q})^Q)$ for the resulting commutative subalgebra of $\mathcal{S}(\mathfrak{q})$.

It was proved in [9] that if $\mathfrak{q} = \mathfrak{g}$ is semisimple and $\xi \in \mathfrak{g}^* \simeq \mathfrak{g}$ is regular semisimple, then $\mathcal{F}_\xi(\mathcal{S}(\mathfrak{g})^G)$ is of maximal dimension. (Later on, it was realised that these subalgebras are also maximal [23].) Let $\mathfrak{q}^*_{reg}$ be the set of Q-regular elements of $\mathfrak{q}^*$. By [2], if $\operatorname{trdeg}(\mathcal{S}(\mathfrak{q})^Q) = \operatorname{ind}\mathfrak{q}$ and $\operatorname{codim}(\mathfrak{q}^* \setminus \mathfrak{q}^*_{reg}) \geqslant 2$, then $\mathcal{F}_\xi(\mathcal{S}(\mathfrak{q})^Q)$ is of maximal dimension for any $\xi \in \mathfrak{q}^*_{reg}$. In [18], we extended this result by proving that the maximality of $\mathcal{F}_\xi(\mathcal{S}(\mathfrak{q})^Q)$ is related to the property that $\operatorname{codim}(\mathfrak{q}^* \setminus \mathfrak{q}^*_{reg}) \geqslant 3$, see also Theorem 1.4 for the precise statement.

An important class of non-reductive Lie algebras consists of $\mathbb{Z}_2$-contractions of semisimple Lie algebras $\mathfrak{g}$. A *$\mathbb{Z}_2$-contraction* of $\mathfrak{g}$ is the semi-direct product $\mathfrak{k} = \mathfrak{g}_0 \ltimes \mathfrak{g}_1$, where $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is a $\mathbb{Z}_2$-grading of $\mathfrak{g}$ and $\mathfrak{g}_1$ becomes an abelian ideal in $\mathfrak{k}$. All $\mathbb{Z}_2$-contractions $\mathfrak{k}$ satisfy the property that $\operatorname{trdeg}(\mathcal{S}(\mathfrak{k})^K) = \operatorname{ind}\mathfrak{k}$ and $\operatorname{codim}(\mathfrak{k}^* \setminus \mathfrak{k}^*_{reg}) \geqslant 2$ [14]. In this paper, we attack the problem of describing maximal commutative subalgebras in $\mathcal{S}(\mathfrak{k})$. Our first approach is to verify when Theorem 1.4 applies to $\mathfrak{k}$. To this end, it suffices to check that $\mathfrak{k}^* \setminus \mathfrak{k}^*_{reg} \geqslant 3$. Our main result is that such "*codim*-3 property" can be characterised in terms of the Satake diagram of the corresponding involution of $\mathfrak{g}$. Namely, $\mathfrak{k} = \mathfrak{g}_0 \ltimes \mathfrak{g}_1$ has the *codim*-3 property if and only if each white node of the Satake diagram has either a black adjacent node or an arrow attached (Theorem 4.1). See also Table 1 for the list of relevant involutions and Satake diagrams. Thus, for those $\mathbb{Z}_2$-contractions, the commutative subalgebras $\mathcal{F}_\xi(\mathcal{S}(\mathfrak{k})^K)$, $\xi \in \mathfrak{k}^*_{reg}$, are maximal. Quite a different approach works if $\mathfrak{g}_1$ contains a regular nilpotent element of $\mathfrak{g}$. Here we prove that $\mathcal{S}(\mathfrak{k})^{\mathfrak{g}_1}$ is a maximal commutative subalgebra (Theorem 3.3). An interesting feature is that, for all cases, both constructions provide maximal commutative subalgebras of $\mathcal{S}(\mathfrak{k})$ that are polynomial. Unfortunately, these results do not cover all $\mathbb{Z}_2$-contractions. On the other hand, there is an involution of $\mathfrak{g} = \mathfrak{sl}_{2n+1}$, with $\mathfrak{g}_0 = \mathfrak{sl}_n \dotplus \mathfrak{sl}_{n+1} \dotplus \mathfrak{t}_1$, where both approaches apply and the resulting commutative subalgebras appear to be rather different.

In Section 1, we gather basic facts on coadjoint representations and commutative subalgebras of $\mathcal{S}(\mathfrak{q})$, including our sufficient condition for the maximality of subalgebras of the form $\mathcal{F}_\xi(\mathcal{S}(\mathfrak{q})^Q)$. Necessary background on the isotropy representations of symmetric spaces and Satake diagrams is presented in Section 2. In Section 3, we recall basic properties of $\mathbb{Z}_2$-contractions and prove that the subalgebra $\mathcal{S}(\mathfrak{k})^{\mathfrak{g}_1}$ is maximal and commutative if and only if $\mathfrak{g}_1$ contains a regular nilpotent element of $\mathfrak{g}$. Section 4 is devoted to our characterisation of the involutions of semisimple Lie algebras having the property that $\operatorname{codim}(\mathfrak{k}^* \setminus \mathfrak{k}^*_{reg}) \geqslant 3$. Finally, in Section 5, we summarise our knowledge on maximal commutative subalgebras of $\mathbb{Z}_2$-contractions and pose open problems.

Notation. If Q acts on an irreducible affine variety X, then $\Bbbk[X]^Q$ is the algebra of Q-invariant regular functions on X and $\Bbbk(X)^Q$ is the field of Q-invariant rational functions. If $\Bbbk[X]^Q$ is finitely generated, then $X/\!\!/Q := \operatorname{Spec}\Bbbk[X]^Q$, and the *quotient morphism* $\pi : X \to X/\!\!/Q$ is the mapping associated with the embedding $\Bbbk[X]^Q \hookrightarrow \Bbbk[X]$.