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Distribution functions, extremal limits and optimal transport

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Dedicated to the 125th anniversary of J.G. van der Corput

Abstract

Encouraged by the study of extremal limits for sums of the form

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} c(x_n, y_n)$$

with uniformly distributed sequences $\{x_n\}$, $\{y_n\}$ the following extremal problem is of interest

$$\max_{\gamma} \int_{[0,1]^2} c(x, y)\gamma(dx, dy),$$

for probability measures γ on the unit square with uniform marginals, i.e., measures whose distribution function is a copula.

The aim of this article is to relate this problem to combinatorial optimization and to the theory of optimal transport. Using different characterizations of maximizing γ's one can give alternative proofs of some results from the field of uniform distribution theory and beyond that treat additional questions. Finally, some applications to mathematical finance are addressed.

Keywords: Distribution functions; Optimal transport; Linear assignment problem; Copulas

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1. Introduction and motivation

In a series of papers J.G. van der Corput [47,48] systematically investigated distribution functions of sequences of real numbers. Some of his main results are as follows:

(i) Any sequence of real numbers has a distribution function.
(ii) Any everywhere dense sequence of real numbers can be rearranged in such a way that the new sequence has an arbitrarily given distribution function.

Clearly, in general a distribution function is not uniquely determined by the sequence. Furthermore, van der Corput established necessary and sufficient conditions for a set $\mathcal{M}$ of non-decreasing functions such that $\mathcal{M}$ is the set of distribution functions of some sequence of real numbers.

More recently, the study of distribution functions was extended to multivariate functions by the Slovak school of O. Strauch and his coworkers; see [5,6,24,44]. In particular, they studied properties of the set of distribution functions of sequences in $[0, 1]^2$ and various extremal problems related to distribution functions. It should be noted that bi-variate distribution functions are well-known in financial mathematics for modeling dependencies in risk processes.

From Fialová and Strauch [23, Theorem 1] one knows that for uniformly distributed sequences $\{x_n\}$, $\{y_n\}$ in $[0, 1]$ and a continuous function $c : [0, 1]^2 \to \mathbb{R}$ one has, along a subsequence $\{N_i\}_{i\in\mathbb{N}}$,

$$\lim_{N_i\to\infty} \frac{1}{N_i} \sum_{n=1}^{N_i} c(x_n, y_n) = \int_{[0,1]^2} c(x, y)\gamma(dx, dy), \tag{1}$$

where γ is a probability measure on the unit square equipped with the σ-algebra of Borel sets. Such a measure exhibits a bi-variate distribution function $C : [0, 1]^2 \to [0, 1]$ which is generally called a *copula*. The aim of this article is to provide a connection between the problem of finding extremal limits in (1) (or maximal and minimal bounds for such limits) by studying the optimization problem

$$\int_{[0,1]^2} c(x, y)\gamma(dx, dy) \mapsto \max \tag{2}$$

and the field of optimal transport. Indeed, we will show how this problem can be perfectly embedded in the general theory of optimal transport.

Motivated by the discussion on the limiting property (1) problem (2) attracted some attention in the number theoretic community and found its way on the collection of unsolved problems of *Uniform Distribution Theory*.[1] We will mention some existing results in that context below. Notice that in the uniform distribution literature, problem (2) is originally written as an optimization with respect to functions $C: [0, 1]^2 \to [0, 1]$ satisfying the following properties: for every $x, y \in [0, 1]$

$$C(x, 0) = C(0, y) = 0,$$
$$C(x, 1) = x \quad \text{and} \quad C(1, y) = y,$$

[1] Problem 1.29 in the open problem collection as of 28. November 2013 (http://www.boku.ac.at/MATH/udt/unsolvedproblems.pdf).

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