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A stable and accurate Davies-like relaxation procedure using multiple penalty terms for lateral boundary conditions

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ABSTRACT

A lateral boundary treatment using summation-by-parts operators and simultaneous approximation terms is introduced. The method is similar to Davies relaxation technique used in the weather prediction community and have similar areas of application, but is also provably stable. In this paper, it is shown how this technique can be applied to the shallow water equations, and that it reduces the errors in the computational domain.

1. Introduction

Accurate numerical calculations on large domains are often unfeasible. To reduce the computational effort, a coarse global mesh is often used with inter-spaced local domains. The local domains are typically employed where fine meshes are needed to capture phenomena with higher accuracy than in the global domain. In the weather prediction community, local domains are often used to model local weather phenomena.

The local domains need to be matched correctly to the global one. Davies (1976) introduced a lateral boundary procedure for weather prediction models where the numerical results on the coarse grid are interpolated into the local, fine grid domain. The method was later referred to as Davies relaxation. Other types of interpolation methods are also used within the weather prediction community, see Williamson and Browning (1974), Birchfield (1960) and Hill (1968). In most applications, one aims for accurate results in the local domains, while the accuracy in the global domain is considered given. Therefore, the coupling between the domains is in most cases neglected. There are methods, see Harrison and Elsberry (1972), where the coupling between the meshes is considered, such that the results from the local domain influence the global one. However, this coupling often introduces stability issues unless optimally done, see for example Nordström et al. (2009b,a). In this work, we will only consider the case where the domains are uncoupled, which is known as one-way nesting.

Many existing lateral boundary schemes produce reflections at the lateral boundaries and experience issues due to over-specification of boundary data (Davies, 1983). These issues are commonly dealt with by choosing the relaxation coefficients properly (Lehmann, 1993) and adding dissipation to the numerical scheme (Davies, 1983). However, altering the scheme in

Abbreviations: SBP, summation-by-parts; SAT, simultaneous approximations terms; MPT, multiple penalty technique.

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these manners may result in an inaccurate and even unstable scheme. The method introduced in this work can effectively reduce reflections at the lateral boundaries (Nordström et al., 2014), without ruining stability and accuracy.

We demonstrate our technique on the shallow water equations, discretized using operators with the summation-by-parts (SBP) property (Carpenter et al., 1999; Strand, 1994; Mattsson and Nordström, 2004), augmented with simultaneous approximation terms (SAT) (Carpenter et al., 1994). This is often referred to as the SBP-SAT technique, and a comprehensive review is given by Svärd and Nordström (2014). If well-posed boundary conditions are available, this technique yields stable numerical schemes (Carpenter et al., 1999).

Nordström et al. (2014) showed that if data is available inside the computational domain, additional penalty terms can be applied without ruining stability. This method, which we call the multiple penalty technique (MPT), can be used to assimilate global data into local area models, construct non-reflecting boundaries, improve accuracy of the scheme and increase the rate of convergence to steady state, see also Erickson and Nordström (2014). The MPT is similar to the Davies relaxation technique mentioned above, but can also be proven to be stable.

The rest of the paper will proceed as follows. In the following section, we define well-posedness of a problem and stability of a numerical scheme. Section 3 describes the SBP-SAT technique and MPT on a simple model problem in one space dimension. In Section 4, we derive well-posed boundary conditions for the shallow water equations and discretize them by using the SBP-SAT technique. Stability conditions are derived and the MPT is applied such that stability is preserved. In Section 5, numerical experiments are performed to illustrate the increased rate of convergence when using the MPT. The assessments of this paper are concluded in Section 6.

2. Preliminaries

Before moving on to the problem under consideration, we will define well-posedness of an initial-boundary value problem (IBVP) and stability of a numerical approximation.

Consider the problem

$$\begin{aligned} \frac{\partial u}{\partial t} &= Hu + F, \quad \bar{r} \in \Omega, t \geq 0 \\ u(\bar{r}, 0) &= f(\bar{r}) \\ Lu(\bar{r}, t) &= g(t), \quad \bar{r} \in \delta\Omega \end{aligned} \tag{1}$$

where H is a differential operator, Ω the spatial domain, $\delta\Omega$ the boundary to Ω, $\bar{r}$ the position vector and F, f, g are known functions. In (1), L is a boundary operator. The problem (1) is well-posed if it has a unique solution that, for $g = F = 0$, satisfies

$$||u(\bar{r}, t)||^2 \leq K_c^1 ||f||^2, \tag{2}$$

where K_c^1 is bounded for finite time and independent of f. Moreover, (1) is strongly well-posed if

$$||u(\bar{r}, t)||^2 \leq K_c^2 [||f||^2 + \int_0^t (||F||^2 + ||g||^2) d\tau] \tag{3}$$

for non-zero g and F. In (2) and (3), $||f||$, $||g||$ and $||F||$ can be expressed in arbitrary norms. The function K_c^2 is bounded for finite time and does not depend on f, g and F.

Let a global semi-discrete approximation of (1) be

$$\begin{aligned} \frac{\partial \mathbf{v}}{\partial t} &= \mathbf{H}\mathbf{v} + \mathbf{F}, \quad t \geq 0 \\ \mathbf{v}(0) &= \mathbf{f} \\ \mathbf{L}\mathbf{v}(t) &= \mathbf{g} \end{aligned} \tag{4}$$

where $\mathbf{v}$ is the numerical approximation of u and $\mathbf{H}$ is a discrete operator that approximates H. The functions $\mathbf{F}$, $\mathbf{f}$, $\mathbf{g}$ are grid functions of F, f and g, i.e. the values are injected at the grid points. Analogously to (2), (4) is stable if

$$||\mathbf{v}(t)||^2 \leq K_d^1 ||\mathbf{f}||^2 \tag{5}$$

for $\mathbf{F} = \mathbf{g} = 0$. In (5), K_d^1 is bounded for finite time and independent of $\mathbf{f}$ and the mesh size. For non-zero $\mathbf{g}$ and $\mathbf{F}$, we say that (4) is strongly stable if

$$||\mathbf{v}(t)||^2 \leq K_d^2 \left[||\mathbf{f}||^2 + \int_0^t (||\mathbf{F}||^2 + ||\mathbf{g}||^2) d\tau \right] \tag{6}$$

where K_d^2 is bounded for finite time and independent of **f, g, F** and the mesh-size. As in the continuous estimate, **||f||, ||g||** and **||F||** may be given in arbitrary norms. For a detailed discussion of well-posedness and stability, the reader is referred to Svärd and Nordström (2014).