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# Splitting phenomenon of a higher-order shallow water theory associated with a longitudinal planetary waves

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ABSTRACT

The Cauchy–Poisson free boundary problem associated with a nonstationary motion of a perfect incompressible fluid circulating around the equatorial plane of a planet is considered. It is shown that the corresponding theory of a higher-order shallow approximation admits two functionally independent systems, while the classical problem for the flat bottom admits only one system.



## 1. Introduction

The large-scale atmospheric dynamics is usually described by moving air masses on a sphere or circle in terms of three and two dimensional Navier–Stokes or Euler equations a thin rotating spherical shell (Blinova, 1943, 1956; Lions et al., 1992, 1992; Ibragimov and Ibragimov, 2011; Ibragimov, 2011; Ibragimov and Pelinovsky, 2009, 2007; Ibragimov et al., 2014, 2012, 2013) or within the theory of shallow water approximation (Iftimie and Raugel, 2001; Weijer et al., 2007; Ibragimov and Villasenor, 2014; Ibragimov, 2010; Okamoto, 1986). The modeling of such moving air masses plays an important role in understanding the global climate control (Summerhayes and Thorpe, 1996; Anderson et al., 2009; Boehm and Lee, 2003). The inclusion of a spherical shape on global scales creates a cyclonic rotation around the poles, i.e., west-to-east winds (Bachelor, 1967; Ibragimov and Pelinovsky, 2009). Namely, as has been indicated already in the late 1800s by Herrmann (1896), the temperature difference between the equator and the poles of a sphere gives rise to waves of two kinds. The first kind consists of waves that advance in the direction of the meridian; the second kind includes equatorial waves. The atmospheric pressures and motions resulting from the combination of these two groups of intersecting waves give rise to the cyclonic and anticyclonic phenomena which are nowadays a paramount topic of research in atmospheric modeling.

Permanent water waves have been considered in a large number of papers. However, most researchers are concerned with fluid motion which is infinitely deep and extends infinitely both rightward and leftward (see e.g. (Crapper, 1984; Stoker, 1957) or (Stokes, 1847) for the history). Such problems are usually called Stokes's problem if the surface tension is neglected and Wilton's problem if the surface tension is taken into account (see e.g. (Okamoto, 1986)). In the first approximation, shallow water equations represent the mathematical theory that can be used to investigate the fluid flows in channels (see e.g. (Friedrichs and Hyers, 1954)). However, this theory does not reveal the role of an undisturbed level of the fluid surface which is needed to determine the precision of the first approximation. A higher order approximation is derived in this work.

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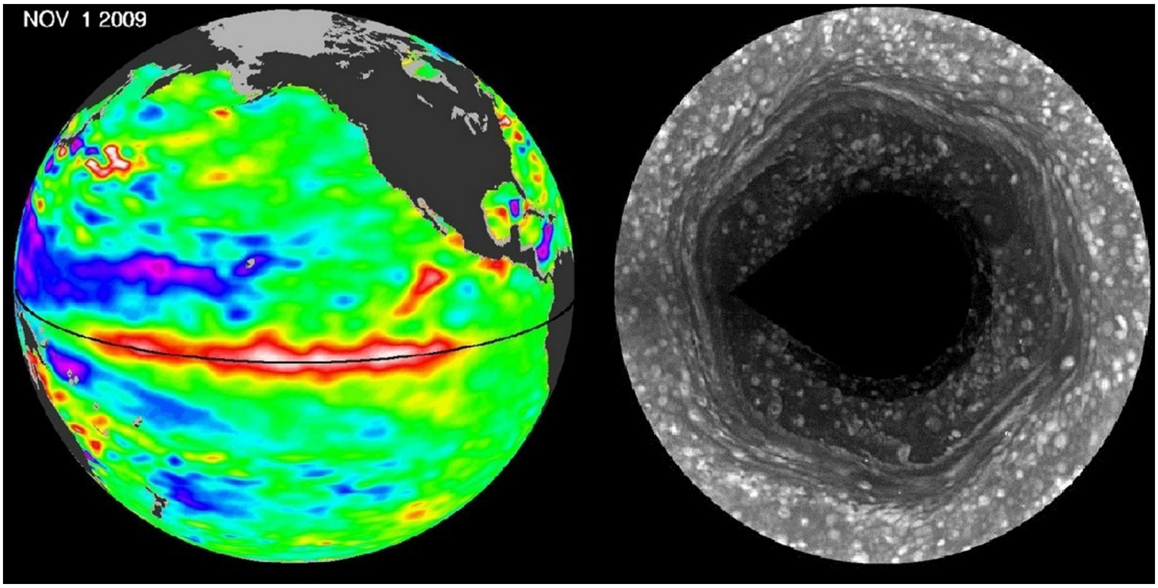


**Fig. 1.** Left: Sea-level height data from November 2009 showing the dynamics of warm water known as Kelvin waves that can be seen traveling eastward along the equator (black line) in Nov. 01, 2009 image. El Ninos form when trade winds in the equatorial western Pacific relax over a period of months, sending Kelvin waves eastward across the Pacific like a conveyor belt. Image credit: NASA/JPL. Right: Image from Cassini, made possible only as Saturn's north pole emerged from winter darkness, shows new details of a jet stream that follows a hexagon-shaped path and has long puzzled scientists.

Here we derive a shallow water equations modeling a simple longitudinal atmospheric motion. For the sake of simplicity, the motion of the fluid is supposed to be irrotational and pressure on a free boundary is constant. It is postulated that the fluid depth is small compared to the radius of the circle and the gravity vector is directed to the center of the circle. As has been discussed in Ibragimov (2001, 2010), under these assumptions, this problem can be associated with an atmospheric longitudinal circulation around equatorial plane.

Such model can be associated e.g. with a large-scale eastward moving, wave of warm water, known as a Kelvin wave that can be seen traveling eastward along the equator as shown in the left panel of Fig. 1. Particularly, in the central and eastern equatorial Pacific, the warm wave appears as the large area of higher-than-normal sea surface heights – warmer-than-normal sea surface temperatures – between 170 degrees east and 100 degrees west longitude.

Another spectacular example of circulating waves is domesticated on the left panel of Fig. 1 showing a jet stream that follows a hexagon-shaped path at the north pole of Saturn. The hexagon was originally discovered in images taken by Voyager spacecraft in the early 1980s. This image also shows another unexplained phenomena such as waves that can be seen traveling along hexagon. These waves and the six-sided shape of the jet stream remain a mystery up to the date.

In earlier work by Ibragimov and Pelinovsky (2009) and Ibragimov (2011), the exact solutions associated with an incompressible viscous and non-viscous fluid flows in a thin spherical shell were found and investigated. Here we aim to investigate the shallow water model associated with long waves formed in the equatorial plane. It is shown that at the higher order approximation the splitting phenomenon of shallow water theory holds, i.e. the model admits two different systems of shallow water equations, which was not observed for a classical problem for the flat bottom. Additionally, the singular exact solutions are provided for the exact mathematical model describing longitudinal planetary waves. Computational experiments Iftimie and Raugel (2001), Ben-Yu (1995), Swarztrauber (2004) and Weijer et al. (2007) provide a credible evidence to support the assertion that singular solutions to the shallow water equations may exist on a stationary sphere. They also suggest that singular solutions are less likely on a rotating sphere. However, the experiments conducted up to the date are not sufficiently extensive to support any credible evidence on the existence or nonexistence of singular solutions to the shallow water equations on a rotating sphere. From the practical standpoint, it is useful to note that the fluid particles at the North and South Poles spin around themselves at a rate $\Omega = 2\pi$ rad/day, whereas fluid particles in the polar (latitude) domain $\theta \in \left[\theta_0, \pi - \theta_0\right]$ do not spin around themselves but simply translate provided $\theta_0 \in (0, \frac{\pi}{2})$. Thus the achievable meteorological flows rotating around the poles correspond to the flows that are being translated along the equatorial plane.

## 2. The model

We introduce polar coordinates $x = r\cos\theta$, $y = r\sin\theta$ and use the following notation: $R$ is the radius of the Earth, $\theta$ is a polar angle, $r$ is the distance from the origin, $h = h_0 + \eta\left(t, \theta\right)$, where $h_0$ is undisturbed level of atmosphere above the Earth and $\eta\left(t, \theta\right)$ is the level of disturbance of a free boundary, as shown schematically in Fig. 2. It is supposed in what follows that $\theta \in [0, 2\pi]$ while $r \in \left[R, h\left(t, \theta\right)\right]$. The homogeneous gravity field $\vec{g}$ is assumed to be a constant and directed to the center