Journal of Structural Geology 56 (2013) 142–155

Contents lists available at ScienceDirect

Journal of Structural Geology

journal homepage: www.elsevier.com/locate/jsg

# Non-steady homogeneous deformations: Computational techniques using Lie theory, and application to ellipsoidal markers in naturally deformed rocks

Joshua R. Davis [a,*], Sarah J. Titus [b], Eric Horsman [c]

[a] Department of Mathematics, Carleton College, 1 N. College St., Northfield, MN 55057, USA
[b] Department of Geology, Carleton College, USA
[c] Department of Geological Sciences, East Carolina University, USA

ARTICLE INFO

*Article history:*
Received 29 January 2013
Received in revised form
16 August 2013
Accepted 17 August 2013
Available online 5 September 2013

*Keywords:*
Velocity gradient
Deformable ellipsoid
Viscous flow
Lie group
Inverse model

ABSTRACT

The dynamic theory of deformable ellipsoidal inclusions in slow viscous flows was worked out by J.D. Eshelby in the 1950s, and further developed and applied by various authors. We describe three approaches to computing Eshelby's ellipsoid dynamics and other homogeneous deformations. The most sophisticated of our methods uses differential-geometric techniques on Lie groups. This Lie group method is faster and more precise than earlier methods, and perfectly preserves certain geometric properties of the ellipsoids, including volume. We apply our method to the analysis of naturally deformed clasts from the Gem Lake shear zone in the Sierra Nevada mountains of California, USA. This application demonstrates how, given three-dimensional strain data, we can solve simultaneously for best-fit bulk kinematics of the shear zone, as well as relative viscosities of clasts and matrix rocks.


## 1. Introduction

Geologists often use elliptical and ellipsoidal markers to characterize strain and infer rheology. Depending on the structural or tectonic context, the ratio $r$ between a clast's viscosity and the host rock's viscosity may vary widely. Passive markers, where $r = 1$, are perhaps the most thoroughly understood case (e.g., Ramsay, 1967; Dunnet, 1969; Elliott, 1970; Matthews et al., 1974; Lisle, 1977). Examples include reduction spots (e.g., Tullis and Wood, 1975) and ooids (e.g., Cloos, 1947, 1971). Rigid clasts, where $r = \infty$, are another important special case. Jeffery (1922) developed a dynamic theory of rigid ellipsoid rotation, which has been applied extensively (Gay, 1968a, b; Ghosh and Ramberg, 1976; Passchier, 1987; De Paor, 1988; Jezek et al., 1996; Simpson and De Paor, 1997; Jessup et al., 2007). Voids, where $r = 0$, represent the other end-member case. Voids have been used in volcanology to study the kinematics of flowing lavas (e.g., Rust et al., 2003) and to estimate magma viscosity (e.g., Manga et al., 1998).

However, ellipsoidal markers in rocks are not always well-modeled by these three idealized special cases. Competent clasts may exhibit viscosity ratios $1 < r < \infty$, and incompetent clasts may exhibit $0 < r < 1$ (Fig. 1). A dynamic theory of deformable ellipsoids in slow viscous flows was developed by Eshelby (1957, 1959) and Bilby et al. (1975). This theory handles all viscosity ratios, including the special cases of passive, rigid, and void ellipsoids. The theory has been extended with new computational approaches (Freeman, 1987; Schmid and Podladchikov, 2003; Mulchrone and Walsh, 2006; Jiang, 2007a), and modified to handle interacting clasts (Mandal et al., 2003), different clast or matrix properties (e.g., Fletcher, 2004, 2009; Dabrowski and Schmid, 2011; Mancktelow, 2011), and other clast shapes (Treagus and Treagus, 2001; Treagus, 2002).

In this paper, we present new approaches to computing Eshelby's deforming ellipsoids. Our main method relies on a mathematical structure called a *Lie group* (see, e.g., Belinfante and Kolman, 1972; Gilmore, 1974; Curtis, 1984; Hall, 2003; Pollatsek, 2009), explained in more detail in Section 3.3. Like earlier methods, our method produces a numerical approximation to the deformation of the ellipsoids, but with three advantages. First, the method automatically preserves desirable characteristics of the ellipsoids, such as their volumes and basic ellipsoidal shape.

* Corresponding author. Tel.: +1 507 645 4402.
*E-mail addresses:* jdavis@carleton.edu (J.R. Davis), stitus@carleton.edu (S.J. Titus), horsmane@ecu.edu (E. Horsman).

0191-8141/$ – see front matter © 2013 Elsevier Ltd. All rights reserved.
http://dx.doi.org/10.1016/j.jsg.2013.08.003

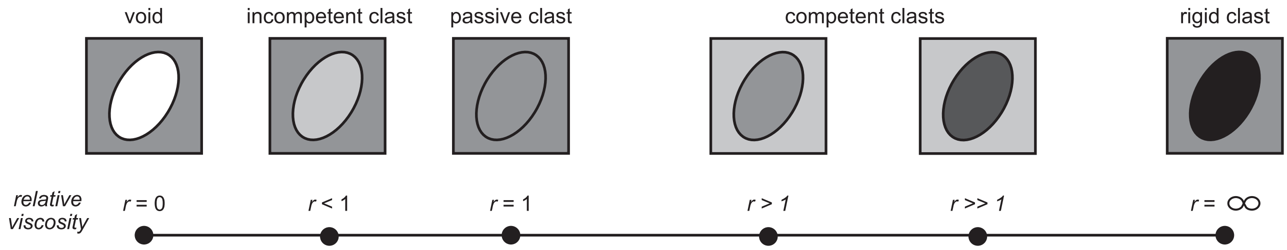


**Fig. 1.** The spectrum of possible ellipsoid problems, organized by viscosity ratio *r*.

Second, the method applies not just to Eshelby's ellipsoids, but to all non-steady homogeneous deformations. Third, the method is faster and more precise than earlier methods. Such speed is of practical value to geologists, because the analysis of naturally deformed rocks may require the simulation of many ellipsoid deformations. As an example application, we inverse-model deformed clast data from the Gem Lake shear zone in the Sierra Nevada mountains of California, USA (Section 5). This application demonstrates how, given three-dimensional strain data, we can solve simultaneously for best-fit bulk kinematics of the shear zone as well as viscosity ratios of several clast types. We compare our results with those from a previously published study that relied on traditional strain modeling.

## 2. Mathematical framework

Consider a rock that contains a single ellipsoidally shaped inclusion of a different viscosity. We subject the rock to a volume-preserving homogeneous deformation. As the matrix rock deforms, the inclusion deforms differently due to the viscosity contrast. We assume that both the matrix and the inclusion materials remain homogeneous, isotropic, and of constant viscosity at all times.

In coordinates $\mathbf{x} = [x_1\, x_2\, x_3]^\top$ centered on the inclusion (Fig. 2), the velocity $\dot{\mathbf{x}}$ at each point in the host rock is linearly related to the position vector $\mathbf{x}$ at that point by a velocity gradient tensor $\mathbf{L}$:

$$\dot{\mathbf{x}} = \mathbf{L}\mathbf{x}. \tag{1}$$

We are denoting by $\mathbf{L}$ both the tensor and its matrix representation in the $\mathbf{x}$ coordinates. For simplicity of presentation, we assume that the deformation is steady, so that $\mathbf{L}$ is constant. The techniques of this paper could be extended to time-dependent $\mathbf{L}$ easily. Because the flow preserves volume, $\operatorname{tr}\mathbf{L} = 0$. There are no other assumptions or restrictions on $\mathbf{L}$.

At any given time, the boundary ellipsoid of the inclusion can be described by a tensor $\mathbf{E}$ (e.g., Flinn, 1979), in that the ellipsoid is the set of points $\mathbf{x}$ such that

$$\mathbf{x}^\top \mathbf{E}\mathbf{x} = 1. \tag{2}$$

Mathematically, $\mathbf{E}$ is a symmetric, positive-definite (0, 2)-tensor. Symmetry and positive-definiteness mean that Eq. (2) defines an ellipsoid, rather than a hyperboloid or other unphysical shape, and that $\mathbf{E}$ diagonalizes as $\mathbf{E} = \mathbf{Q}^\top \tilde{\mathbf{E}}\, \mathbf{Q}$, where $\mathbf{Q}$ is a rotation matrix, and the $a_i > 0$ (Fig. 3). The rows of $\mathbf{Q}$ are unit vectors (in $\mathbf{x}$ coordinates) indicating the directions of the ellipsoid semi-axes in a right-handed manner, and the $a_i$ are the semi-axis lengths. The matrix $\tilde{\mathbf{E}}$ is the tensor $\mathbf{E}$ rendered in a coordinate system $\tilde{\mathbf{x}} = [\tilde{x}_1 \tilde{x}_2 \tilde{x}_3]^\top$ aligned with the ellipsoid's axes (Fig. 2). When two of the semi-axis lengths $a_i$ and $a_j$ are equal, there is an apparent ambiguity in choosing the $\tilde{\mathbf{x}}$ coordinates, or equivalently $\mathbf{Q}$. However, there is a unique correct way to resolve this ambiguity (see Jiang (2007a) and Appendix B). The volume of the ellipsoid is

$$\tilde{\mathbf{E}} = \begin{bmatrix} a_1^{-2} & 0 & 0 \\ 0 & a_2^{-2} & 0 \\ 0 & 0 & a_3^{-2} \end{bmatrix}, \tag{3}$$

$$\frac{4}{3}\pi a_1 a_2 a_3 = \frac{4\pi}{3\sqrt{\det \mathbf{E}}}.$$

Under the assumption of slow viscous flow, Eshelby (1957) and Bilby et al. (1975) showed that, if the inclusion is an ellipsoid, then its deformation is also homogeneous. That is, for all $\mathbf{x}$ inside the inclusion and along its boundary,

$$\dot{\mathbf{x}} = \mathbf{K}\mathbf{x}, \tag{4}$$

for some velocity gradient tensor $\mathbf{K}$ (viewed as a matrix, in the $\mathbf{x}$ coordinates). The dynamics of the ellipsoid are most easily described in the $\tilde{\mathbf{x}}$ coordinates (Appendix A). As the ellipsoid deforms, $\det \mathbf{E}$ and the ellipsoid volume remain constant, but $\mathbf{K}$ continually changes. Thus Eshelby's theory is an example of a non-steady homogeneous deformation. In contrast to the steady homogeneous case (Provost et al., 2004; Davis and Titus, 2011), the differential equation that governs the non-steady case (Eq. (4)) admits no closed-form solution. One must resort to an iterative algorithm that produces a numerical approximation to the deformation. An inherent trade-off exists between the precision of the simulation and the computational time required. Imprecision causes the ellipsoid to drift away from its true size, shape, and orientation. In some cases, imprecision may lead to a catastrophic failure, such as an ellipsoid that degenerates to a cylinder or

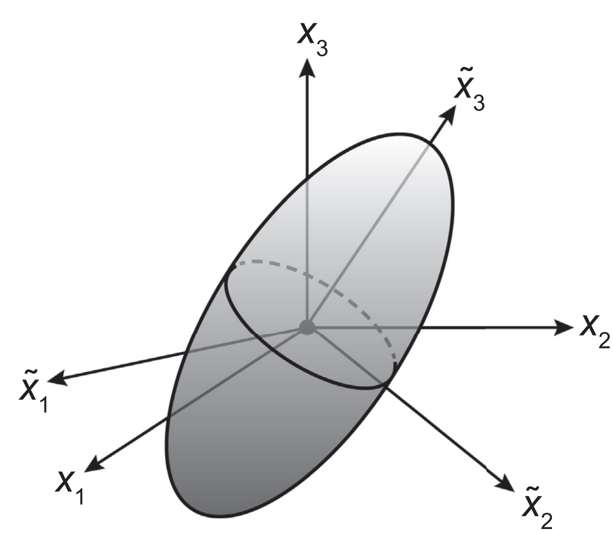


**Fig. 2.** Two coordinate systems are employed. The $\mathbf{x}$ coordinates are fixed. The $\tilde{x}$ coordinates are aligned with the axes of the ellipsoid, and rotate over time.